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Exit time and invariant measure asymptotics for small noise constrained diffusions

Anup Biswas[a,*], Amarjit Budhiraja[b]

[a] Centre for Applicable Mathematics, Tata Institute of Fundamental Research, Post Bag no 03, Chikkabommasandra, Bangalore-560065, India
[b] Department of Statistics and Operations Research, University of North Carolina, Chapel Hill, NC 27599, USA

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Abstract

Constrained diffusions, with diffusion matrix scaled by small $\epsilon > 0$, in a convex polyhedral cone $G \subset \mathbb{R}^k$, are considered. Under suitable stability assumptions small noise asymptotic properties of invariant measures and exit times from domains are studied. Let $B \subset G$ be a bounded domain. Under conditions, an "exponential leveling" property that says that, as $\epsilon \to 0$, the moments of functionals of exit location from B, corresponding to distinct initial conditions, coalesce asymptotically at an exponential rate, is established. It is shown that, with appropriate conditions, difference of moments of a typical exit time functional with a sub-logarithmic growth, for distinct initial conditions in suitable compact subsets of B, is asymptotically bounded. Furthermore, as initial conditions approach 0 at a rate ϵ^2 these moments are shown to asymptotically coalesce at an exponential rate.

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1. Introduction

Diffusions in polyhedral domains arise commonly as approximate models for stochastic processing networks in heavy traffic [26,28,23,11]. In this work we consider a family of such

* Corresponding author. Tel.: +91 80 66953756; fax: +91 80 66953799.
E-mail addresses: anup@math.tifrbng.res.in (A. Biswas), budhiraj@email.unc.edu (A. Budhiraja).

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constrained diffusions with a small parameter (denoted as ϵ) multiplying the diffusion coefficient. Goal of this work is the study of asymptotic properties of invariant measures and exit times from suitable domains, as $\epsilon \to 0$.

The classical reference for small noise asymptotics of diffusions in $\mathbb{R}^k$ is the book [20]. The basic object of study in this fundamental body of work is a collection of diffusion processes $\{X^\epsilon\}_{\epsilon>0}$, given as

$$dX^\epsilon(t) = b(X^\epsilon(t))dt + \epsilon\sigma(X^\epsilon(t))dW(t), \qquad X^\epsilon(0) = x, \tag{1.1}$$

where W is a k-dimensional standard Brownian motion and b, σ are suitable coefficients. The stochastic process $X^\epsilon \equiv \{X^\epsilon(t)\}_{0\le t\le T}$, for each $T \ge 0$ can be regarded as a $\mathcal{C}_T = C([0, T] : \mathbb{R}^k)$ (space of continuous functions from $[0, T]$ to $\mathbb{R}^k$ with the uniform topology) valued random variable and under suitable conditions on b, σ one can show that, as $\epsilon \to 0$, X^ϵ converges in probability to ξ which is the unique solution of the ordinary differential equation (ODE)

$$\dot{\xi} = b(\xi), \qquad \xi(0) = x. \tag{1.2}$$

One of the basic results in the field says that for each $T > 0$, as $\epsilon \to 0$, X^ϵ satisfies a large deviation principle (LDP) in $\mathcal{C}_T$, uniformly in the initial condition x in any compact set K, with an appropriate rate function $I_T:\mathcal{C}_T \to [0,\infty]$. This result is a starting point for the study of numerous asymptotic questions for such small noise diffusions. In particular, when the underlying diffusions have suitable stability properties, the above LDP plays a central role in the study of asymptotic properties of invariant measures and exit times from domains (see Chapter 4 of [20]; see also [5] where asymptotic properties of invariant densities are studied). The asymptotics are governed by the "quasi-potential" function V which is determined from the collection of rate functions $\{I_T : T > 0\}$.

The theory developed in [20] has been extended and refined in many different directions. One notable work is [13] which studies the asymptotics of solutions of Dirichlet problems associated with diffusions given by (1.1). To signify the dependence on the initial condition, denote the solution of (1.1) by X^ϵ_x. Let B be a bounded domain in $\mathbb{R}^k$ and K be an arbitrary compact subset of B. Under the assumption that all solutions of the ODE (1.2), with $x = \xi(0) \in B$, converge without leaving B, to a single linearly asymptotically stable critical point, paper [13] shows that with suitable conditions on the coefficients, for all bounded measurable f

$$\sup_{x,y\in K} |\mathbb{E}(f(X^\epsilon_x(\tau^\epsilon_x))) - \mathbb{E}(f(X^\epsilon_y(\tau^\epsilon_y)))|$$

converges to 0 at an exponential rate. Here, $\tau^\epsilon_x = \inf\{t : X^\epsilon_x(t) \in B^c\}$. This property, usually referred to as "exponential leveling", says that although the exit time of the process from the domain approaches ∞, as $\epsilon \to 0$, the moments of functionals of exit location, corresponding to distinct initial conditions, coalesce asymptotically, at an exponential rate. The key ingredient in the proof is the gradient estimate

$$\sup_{x\in K} |\nabla u^\epsilon(x)| \le c\epsilon^{-1/2}, \tag{1.3}$$

where u^ϵ is the solution of the Dirichlet problem on B associated with the diffusion (1.1) with boundary data f.

The goal of the current work is to develop the Freidlin–Wentzell small noise theory for a family of constrained diffusions in polyhedral cones. Let $G \subset \mathbb{R}^k$ be convex polyhedral cones in $\mathbb{R}^k$ with the vertex at the origin given as the intersection of half spaces G_i, $i = 1, 2, \ldots, N$. Let

n_i be the unit vector associated with G_i via the relation

$$G_i = \{x \in \mathbb{R}^k : \langle x, n_i \rangle \geq 0\}.$$

We will denote the set $\{x \in \partial G : \langle x, n_i \rangle = 0\}$ by F_i. With each face F_i we associate a unit vector d_i such that $\langle d_i, n_i \rangle > 0$. This vector defines the *direction of constraint* associated with the face F_i. Precise definition of a constrained diffusion is given in Section 2 (see (2.5)), but roughly speaking such a process evolves infinitesimally as a diffusion in $\mathbb{R}^k$ and is instantaneously pushed back using the oblique reflection direction d_i upon reaching the face F_i. Formally, such a process, denoted once more as X_x^ϵ, can be represented as a solution of a stochastic integral equation of the form

$$X_x^\epsilon(t) = \Gamma\left(x + \int_0^\cdot b(X^\epsilon(s))\mathrm{d}s + \epsilon \int_0^\cdot \sigma(X^\epsilon(s))\mathrm{d}W(s)\right)(t), \tag{1.4}$$

where Γ is the Skorokhod map (see below Definition 2.1) taking trajectories with values in $\mathbb{R}^k$ to those with values in G, consistent with the constraint vectors $\{d_i, i = 1, \ldots, N\}$. Under certain regularity assumptions on the Skorokhod map (see Assumption 2.1) and the usual Lipschitz conditions on the coefficients b and σ, our first result (Theorem 2.1) establishes a (locally uniform) LDP for X_x^ϵ, in $C([0, T] : G)$ for each $T > 0$. This result is the starting point for all exit time and invariant measure estimates obtained in this work.

Stability properties of constrained diffusions in polyhedral domains have been studied in [22,19,9,2,12]. Let

$$\mathcal{C} = \left\{-\sum_{i=1}^N \alpha_i d_i : \alpha_i \geq 0;\ i \in \{1, \ldots, N\}\right\}.$$

Paper [2] shows that under regularity of the Skorokhod map, uniform non-degeneracy of σ and Lipschitz coefficients, if for some $\delta > 0$

$$b(x) \in \mathcal{C}(\delta) = \{v \in \mathcal{C} : \mathrm{dist}(v, \partial\mathcal{C}) \geq \delta\} \quad \text{for all } x \in G, \tag{1.5}$$

then the constrained diffusion X^ϵ is positive recurrent and consequently admits a unique invariant probability measure μ^ϵ. In Theorems 5.1 and 5.2 we study the asymptotic properties of μ^ϵ, as $\epsilon \to 0$. For the setting where the diffusion and drift coefficients are constant, analogous results are obtained in [17].

We also consider asymptotic properties of exit times from a bounded domain $B \subset G$ that contains the origin. One important feature in this analysis is that the stability condition (1.5), in general, does not ensure that the trajectories of the associated deterministic dynamical system

$$\xi = \Gamma(x + \int_0^\cdot b(\xi(s))\mathrm{d}s) \tag{1.6}$$

with initial condition in B will stay within the domain at all times. However, a weaker stability property holds, namely, one can find domains $B_0 \subset B$ such that all trajectories of (1.6) starting in B_0 stay within B at all times. This property is used in Theorem 3.1 and Corollaries 3.1, 3.2 to obtain basic asymptotic results for exit times from suitable domains.

A significant part of this work is devoted to the proof of the exponential leveling property for constrained diffusions. Our main result is Theorem 3.2. We recall that the key ingredient in the proof of such a result for diffusions in $\mathbb{R}^k$ is the gradient estimate (1.3) for solutions of the associated Dirichlet problem. For diffusions in domains with corners and with oblique reflection fields that change discontinuously, there are no regularity (eg. C^1 solutions) results known for the associated partial differential equations (PDE). Our proof of the exponential leveling property (Theorem 3.2 (i)) is purely probabilistic and bypasses all PDE estimates. The main step in the proof is the construction of certain (uniform in ϵ) Lyapunov functions which are then used to construct a coupling of the processes X_x^ϵ, X_y^ϵ with explicit uniform estimates on exponential moments of time to coupling (see Lemma 4.7). The key ingredient in this coupling construction is the minorization condition on transition densities of reflected diffusions (see Condition 3.1). Proposition 3.1 in Section 3 gives one basic setting where such a minorization property holds. Obtaining general conditions under which this estimate on transition densities of reflected diffusions holds is a challenging open problem. The minorization property allows for the construction of a pseudo-atom, for each fixed $\epsilon > 0$, using split chain ideas of Athreya–Ney–Nummelin [3,25]. Coupling based on pseudo-atom constructions has been used by many authors for the study of a broad range of stability and control problems [24,6,7,15], however the current paper appears to be the first to bring these powerful techniques to bear for the study of exit time asymptotics of small noise Markov processes.

As a second consequence of our coupling constructions we show in part (ii) of Theorem 3.2 that difference of moments of an exit time functional with a sub-logarithmic growth corresponding to distinct initial conditions in B_0 is asymptotically bounded. Note that for typical unbounded functionals the associated moments will approach ∞, as $\epsilon \to 0$ thus the result provides a rather non-trivial "leveling estimate". The third important consequence of our approach is given in Proposition 3.2 (ii). This result says that as initial conditions approach 0 at a rate ϵ^2 the corresponding moments of exit time functionals with sub-logarithmic growth asymptotically coalesce at an exponential rate. To the best of our knowledge, estimates analogous to Theorem 3.2 (ii) and Proposition 3.2 (ii) are not known even for the setting of small noise diffusions (with suitable stability properties) in $\mathbb{R}^k$.

These estimates are interesting for one-dimensional models as well. Let X be a one-dimensional reflected Brownian motion, starting from $x \in [0, 1)$, with drift $b \in (-\infty, 0)$ and variance $\sigma^2\epsilon^2 \in (0, \infty)$, given on some probability space $(\Omega, \mathcal{F}, \mathbb{P}_x^\epsilon)$. That is, $\mathbb{P}_x^\epsilon$ a.e.,

$$X(t) = x + bt + \epsilon\sigma B(t) - \inf_{0\le s\le t}\{(x + bs + \epsilon\sigma B(s)) \wedge 0\}, \quad t \ge 0$$

where B is a standard Brownian motion. Let $\tau = \inf\{t : X(t) = 1\}$. Let $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ be a measurable map and let $\varphi_\epsilon(x) = \mathbb{E}_x^\epsilon \psi(\tau)$. Then Proposition 3.2 (ii) for this one-dimensional setting says that if ψ has a sub-logarithmic growth then for some $c, \delta \in (0, \infty)$, as $\epsilon \to 0$,

$$e^{\delta/\epsilon}|\varphi_\epsilon(c\epsilon^2) - \varphi_\epsilon(0)| \to 0. \tag{1.7}$$

If $\psi(t) = t$, $t \ge 0$, then φ_ϵ is the unique solution to the boundary value problem

$$\epsilon^2\sigma^2\varphi_\epsilon'' + b\varphi_\epsilon' = -1, \qquad \varphi_\epsilon'(0) = 0, \qquad \varphi_\epsilon(1) = 0,$$

whose solution is given as $\varphi_\epsilon(x) = \frac{\epsilon^2\sigma^2}{b^2}[e^{\frac{-b}{\epsilon^2\sigma^2}} - e^{\frac{-bx}{\epsilon^2\sigma^2}}] + \frac{1}{b}(1 - x)$. It is easily verified that if $x_\epsilon = c\epsilon^2$, for some $c > 0$, then $|\varphi_\epsilon(x_\epsilon) - \varphi_\epsilon(0)| \to 0$ at a rate ϵ^2, as $\epsilon \to 0$. In particular, the decay is not exponential. Furthermore, for $x \ne y$ in $[0, 1)$, $|\varphi_\epsilon(x) - \varphi_\epsilon(y)| \to \infty$ as $\epsilon \to 0$ (compare with Theorem 3.2 (ii)). For general ψ a similar ODE analysis is less tractable and thus the coupling techniques developed here give a powerful approach (especially in higher dimensions) to the study of such asymptotic estimates.

This paper is organized as follows. In Section 2 we introduce the Skorokhod map and give a precise formulation of a constrained diffusion process. We also present in this section (Theorem 2.1) the basic LDP on $C([0,T]:G)$ for small noise constrained diffusions. A useful alternative expression for the rate function (see (2.7)) is given and the quasi-potential that plays a key role in the asymptotic theory is introduced. Section 3 is devoted to the study of exit time asymptotics. One of the key steps in the proofs of our main results is the coupling property given in Theorem 3.3. This Theorem which is at the heart of our analysis is proved in Section 4. In Section 5 we study the asymptotics of invariant measures. Finally Section 6 collects proofs of some results that are based on adaptations of classical arguments.

The following notation will be used. Closure, complement, boundary and interior of a subset B of a topological space S will be denoted by $\bar{B}$, B^c, ∂B and B^o, respectively. We denote the ball $\{x \in \mathbb{R}^k : |x - x_0| < r\}$ by $\mathbb{B}_r(x_0)$. Given a metric space (S,d), and subsets A, B of S, we will define $\text{dist}(A,B) = \inf_{x\in A, y\in B} d(x,y)$. If $A = \{x\}$ for some $x \in S$, we write $\text{dist}(A,B)$ as $\text{dist}(x,B)$. Denote by $C([0,\infty) : S)$ (resp. $C([0,T] : S)$) the space of continuous functions from $[0,\infty)$ (resp. $[0,T]$) to a metric space S. This space is endowed with the usual topology of uniform convergence on compacts. For $\eta \in C([0,\infty) : \mathbb{R}^k)$, and $t > 0$, define $|\eta|(t) = \sup \sum_{i=1}^n |\eta(t_{i+1}) - \eta(t_i)|$, where sup is taken over all partitions $0 = t_1 < t_2 < \cdots < t_{n+1} = t$. The Borel σ-field on a metric space S will be denoted as $\mathcal{B}(S)$ and the space of all probability measures on S will be denoted by $\mathcal{P}(S)$. Infimum over an empty set, by convention, is taken to be ∞. We will denote by $\kappa, \kappa_1, \kappa_2, \ldots$ generic constants whose values may change from one proof to the next.

2. Preliminaries

We begin with the definition of the Skorokhod map. For $x \in \partial G$ define

$$d(x) = \left\{ v \in \mathbb{R}^k : v = \sum_{i\in \text{In}(x)} \alpha_i d_i;\ \alpha_i \geq 0;\ |v| = 1 \right\},$$

where

$$\text{In}(x) = \{i \in \{1,2,\ldots,N\} : \langle x, n_i\rangle = 0\}.$$

Definition 2.1 (*Skorokhod Problem*). Let $\psi \in C([0,\infty), \mathbb{R}^k)$ be given such that $\psi(0) \in G$. Then $(\phi,\eta) \in C([0,\infty), G) \times C([0,\infty), \mathbb{R}^k)$ solves the Skorokhod Problem (SP) for ψ (with respect to the data $\{(d_i, n_i), i = 1,\ldots,N\}$), if $\phi(0) = \psi(0)$ and if for all $t \in [0,\infty)$: (1) $\phi(t) = \psi(t) + \eta(t)$, (2) $|\eta|(t) < \infty$, (3) $|\eta|(t) = \int_0^t \mathbb{I}_{\{\phi(s)\in\partial G\}} \mathrm{d}|\eta|(s)$, and there exists a Borel measurable function $\gamma : [0,\infty) \to \mathbb{R}^k$ such that $\gamma(t) \in d(\phi(t))$, for $\mathrm{d}|\eta|$ a.e. t and $\eta(t) = \int_0^t \gamma(s)\mathrm{d}|\eta|(s)$, $t \geq 0$.

Let $C_G([0,\infty) : \mathbb{R}^k)$ be the collection of $\psi \in C([0,\infty) : \mathbb{R}^k)$ such that $\psi(0) \in G$. The domain $D \subseteq C_G([0,\infty) : \mathbb{R}^k)$ on which there is a unique solution to the Skorokhod problem we define the Skorokhod map (SM) Γ as $\Gamma(\psi) \doteq \phi$ if (ϕ,η) is the unique solution of the Skorokhod problem posed by ψ. We will make the following assumption on the regularity of the Skorokhod map defined by the data $\{(d_i, n_i) : i = 1,2,\ldots,N\}$.

Condition 2.1. *The Skorokhod map is well defined on all of $C_G([0,\infty) : \mathbb{R}^k)$, that is, $D = C_G([0,\infty) : \mathbb{R}^k)$, and the SM is Lipschitz continuous in the following sense: There exists a* *constant $K \in (0, \infty)$ such that for all $\phi_1, \phi_2 \in C_G([0, \infty) : \mathbb{R}^k)$:*

$$\sup_{0 \le t < \infty} |\Gamma(\phi_1)(t) - \Gamma(\phi_2)(t)| < K \sup_{0 \le t < \infty} |\phi_1(t) - \phi_2(t)|. \tag{2.1}$$

We refer the reader to [21,16,18] for sufficient conditions under which Condition 2.1 holds.

Now we introduce the small noise constrained diffusion process that will be considered in this work. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space on which a filtration $\{\mathcal{F}_t\}_{t \ge 0}$ satisfying the usual hypothesis is given. Let $(W(t), \mathcal{F}_t)$ be a k-dimensional standard Wiener process on the above probability space. Let $\sigma : G \to \mathbb{R}^{k \times k}$, $b : G \to \mathbb{R}^k$ be mappings satisfying the following condition.

Condition 2.2. *There exists $\gamma \in (0, \infty)$ such that*

$$|\sigma(x) - \sigma(y)| + |b(x) - b(y)| \le \gamma |x - y| \quad \forall x, y \in G \tag{2.2}$$

and

$$|\sigma(x)| \le \gamma \quad \forall x \in G. \tag{2.3}$$

Given $\epsilon > 0$, let X^ϵ be the unique strong solution of the following stochastic integral equation:

$$X(t) = \Gamma\left(x + \int_0^{\cdot} b(X(s))\mathrm{d}s + \epsilon \int_0^{\cdot} \sigma(X(s))\mathrm{d}W(s)\right)(t), \quad t \ge 0. \tag{2.4}$$

Existence of strong solutions and pathwise uniqueness for (2.4) is a consequence of the Lipschitz property of the coefficients and of the Skorokhod map (see [16]). It is convenient to have the process X for various initial conditions and values of ϵ to be defined on a common canonical space. Indeed, one can find a filtered measurable space, which we denote again as $(\Omega, \mathcal{F}, \{\mathcal{F}_t\})$, on which is given a family of probability measures, $\{\mathbb{P}_x^\epsilon\}_{x \in G}$, and continuous adapted stochastic processes Z, Y and W such that for all $x \in G$, under $\mathbb{P}_x^\epsilon$, $\{W(t), \{\mathcal{F}_t\}_{t \ge 0}\}$ is a k-dimensional standard Wiener process and (Z, W, Y) satisfy $\mathbb{P}_x^\epsilon$ a.s. the integral equation

$$\begin{aligned} Z(t) &= \Gamma\left(x + \int_0^{\cdot} b(Z(s))\mathrm{d}s + \epsilon \int_0^{\cdot} \sigma(Z(s))\mathrm{d}W(s)\right)(t), \\ &= x + \int_0^t b(Z(s))\mathrm{d}s + \epsilon \int_0^t \sigma(Z(s))\mathrm{d}W(s) + \mathbb{D}Y(t), \quad t \ge 0, \end{aligned} \tag{2.5}$$

where $\mathbb{D} = (d_1, \ldots, d_N)$. Also, for every $\epsilon > 0$, $(Z, \{P_x^\epsilon\}_{x \in G})$ is a strong Markov family (cf. [12]).

For a domain $S \subset \mathbb{R}^k$, let $\mathrm{AC}([0, \infty) : S)$ (resp. $\mathrm{AC}([0, T] : S)$) denote the space of absolutely continuous functions on $[0, \infty)$ (resp. $[0, T]$) with values in S. From the Lipschitz property of the SM it follows that if $\psi \in \mathrm{AC}([0, \infty) : \mathbb{R}^k)$ (with $\psi(0) \in G$) then $\phi = \Gamma(\psi) \in \mathrm{AC}([0, \infty) : G)$. For $T > 0$ and $\phi \in \mathrm{AC}([0, T] : G)$, define

$$S_{0T}(\phi) = \inf_{u \in \mathcal{M}(\phi)} \left\{ \frac{1}{2} \int_0^T |\dot{u}(s)|^2 \mathrm{d}s \right\}, \tag{2.6}$$

where $\mathcal{M}(\phi) = \{u \in \mathrm{AC}([0, T] : \mathbb{R}^k) : \phi(t) = \Gamma(\phi(0) + \int_0^{\cdot} b(\phi(s))\mathrm{d}s + \int_0^{\cdot} \sigma(\phi(s))\dot{u}(s)\mathrm{d}s)(t), t \in [0, T]\}$.

The following result is a consequence of the unique pathwise solvability of (2.4). A sketch is provided in Section 6.

Theorem 2.1. *Suppose that Conditions 2.1 and 2.2 hold. For every $T > 0$ and $x \in G$, the family $(\{Z(t)\}_{0\le t\le T}, \mathbb{P}^{\epsilon}_{x})$ satisfies a large deviation principle (LDP), as $\epsilon \to 0$, in $C([0,T] : G)$ with rate function*

$$I_x(\phi) = \begin{cases} S_{0T}(\phi) & \text{if } \phi \in AC([0,T] : G) \text{ and } \phi(0) = x \\ \infty & \text{otherwise.} \end{cases}$$

Additionally, LDP holds uniformly over $x \in K$ for every compact subset $K \subset G$.

We remark that the function I_x is a "good" rate function, namely it has compact sub-level sets (i.e., for each $M \in (0,\infty)$, the set $\{\phi \in C([0,T] : G) : I_x(\phi) \le M\}$ is compact). The following non-degeneracy condition allows for a simpler representation of the rate function.

Condition 2.3. *For some $c \in (0,\infty)$, $y'(\sigma(x)\sigma'(x))y \ge c|y|^2$ for all $x \in G$ and $y \in \mathbb{R}^k$.*

Note that if $\phi \in \mathrm{AC}([0,T] : G)$ and $u \in \mathcal{M}(\phi)$, then

$$\psi(\cdot) = \phi(0) + \int_0^{\cdot} b(\phi(s))\mathrm{d}s + \int_0^{\cdot} \sigma(\phi(s))\dot{u}(s)\mathrm{d}s$$

is in $\mathrm{AC}([0,T] : \mathbb{R}^k)$ and $\phi = \Gamma(\psi)$. Conversely, if $\psi \in \mathrm{AC}([0,T] : \mathbb{R}^k)$ and $\phi = \Gamma(\psi)$ then

$$u(\cdot) = \int_0^{\cdot} \sigma^{-1}(\phi(s))(\dot{\psi}(s) - b(\phi(s)))\mathrm{d}s$$

is in $\mathcal{M}(\phi)$. From these observations it is easy to deduce the following alternative expression for $S_{0T}(\phi)$ for $\phi \in \mathrm{AC}([0,T] : G)$.

$$S_{0T}(\phi) = \inf_{\{\psi\in \mathrm{AC}([0,T]:\mathbb{R}^k):\psi(0)=\phi(0),\phi=\Gamma(\psi)\}} \int_0^T L(\phi(t), \dot{\psi}(t))\mathrm{d}t, \tag{2.7}$$

where for $\alpha, \beta \in \mathbb{R}^k$,

$$L(\alpha,\beta) = \frac{1}{2}(\beta - b(\alpha))' A^{-1}(\alpha)(\beta - b(\alpha)), \qquad A(\alpha) = \sigma(\alpha)\sigma'(\alpha).$$

The goal of this work is to study asymptotic properties of invariant measures of X^{ϵ} and of exit times of X^{ϵ} from suitable domains. Such properties rely on appropriate stability properties of the underlying zero-noise dynamical system. For constrained diffusions considered here, stability conditions that are natural for applications have been identified in [22,19,9,2]. Recall the cone $\mathcal{C}$ and the set $\mathcal{C}(\delta)$ from the introduction (see (1.5)). The following stability estimate was established in [2] (see Theorem 2.1 therein).

Lemma 2.1. *Let $\psi \in AC([0,\infty) : \mathbb{R}^k)$, $\psi(0) \in G$ and $\phi = \Gamma(\psi)$. Suppose that for some $\delta > 0$, $\dot{\psi}(t) \in \mathcal{C}(\delta)$, a.e. t. Then*

$$|\phi(t)| \le \frac{K^2|\phi(0)|^2}{K|\phi(0)| + \delta t}, \qquad t \in (0,\infty).$$

The above result motivates the next condition that will be used in this work.

Condition 2.4. *There exists a* $\delta \in (0, \infty)$ *such that for all* $x \in G$, $b(x) \in \mathcal{C}(\delta)$.

Conditions 2.1–2.4 will be assumed henceforth and thus explicit reference to these conditions in statements of results will be omitted.

Next we introduce the "quasi-potential" function that plays a key role in this work. For $x \in G$ and $\phi \in C([0, \infty) : G)$, define $\zeta(x, \phi) = \inf\{t \geq 0 : \phi(t) = x\}$. Define $V : G \to [0, \infty]$ by the relation

$$V(x) = \inf_{\psi \in \mathrm{AC}([0,\infty):\mathbb{R}^k),\ \psi(0)=0,\zeta(x,\Gamma(\psi))<\infty} j_{\zeta(x,\Gamma(\psi))}(\Gamma(\psi), \psi), \tag{2.8}$$

where for $T > 0$ and $(\phi, \psi) \in \mathrm{AC}([0, \infty) : G) \times \mathrm{AC}([0, \infty) : \mathbb{R}^k)$, $j_T(\phi, \psi) = \int_0^T L(\phi(t), \dot{\psi}(t))\mathrm{d}t$. It is easily checked that V is a continuous function and $V(x) = 0$ if and only if $x = 0$.

3. Exit time estimates

In this section we study the asymptotic properties of exit time of the process $(Z, \mathbb{P}^\epsilon_x)$, from a suitable domain $B \subset G$, as $\epsilon \to 0$. Throughout G will be endowed with the relative topology inherited from $\mathbb{R}^k$. Let B be a bounded open subset of G. Suppose that $0 \in B$ and $\partial B = \partial \bar{B}$.

For $x \in G$ we denote by $\xi_x \in \mathrm{AC}([0, \infty) : G)$ the unique solution of the integral equation

$$\xi_x(t) = \Gamma\left(x + \int_0^\cdot b(\xi_x(s))\mathrm{d}s\right)(t), \quad t \geq 0. \tag{3.1}$$

Also, denote $\mathcal{S}_x = \{\xi_x(t), t \geq 0\}$. For $\gamma > 0$, let $B_\gamma = \{x \in B : \mathrm{dist}(\mathcal{S}_x, \partial B) \geq \gamma\}$ and let $B_0 = \bigcup_{\gamma>0} B_\gamma$. Let $\tau \equiv \tau(B) = \inf\{t \geq 0 : Z(t) \in B^c\}$. Also, let $V_0 \equiv V_0(B) = \inf_{x\in\partial B} V(x)$.

The following theorem follows on minor modifications of Theorems 4.4.1 and 4.4.2 of [20]. A sketch outlining the main differences in the proof is in Section 6.

Theorem 3.1. (i)

$$\limsup_{\epsilon\to 0} \sup_{x\in B} \epsilon^2 \log \mathbb{E}^\epsilon_x(\tau) \leq V_0, \tag{3.2}$$

and for every $\alpha > 0$,

$$\lim_{\epsilon\to 0} \inf_{x\in B} \mathbb{P}^\epsilon_x(\tau < \exp\{\epsilon^{-2}(V_0 + \alpha)\}) = 1. \tag{3.3}$$

(ii) *For every* $x \in B_0$

$$\liminf_{\epsilon\to 0} \epsilon^2 \log \mathbb{E}_x(\tau) \geq V_0, \tag{3.4}$$

and for every $\alpha > 0$,

$$\lim_{\epsilon\to 0} \mathbb{P}^\epsilon_x(\tau > \exp\{\epsilon^{-2}(V_0 - \alpha)\}) = 1. \tag{3.5}$$

As an immediate consequence of the theorem we have the following.

Corollary 3.1. *For any* $x \in B_0$, $\lim_{\epsilon\to 0} \epsilon^2 \log \mathbb{E}^\epsilon_x(\tau) = V_0$ *and for every* $\alpha > 0$

$$\lim_{\epsilon\to 0} \mathbb{P}^\epsilon_x(|\epsilon^2 \log \tau - V_0| > \alpha) = 0.$$

From Lemma 2.1 we see that if $B = \mathbb{B}_\lambda = \{x \in G : |x| \le \lambda\}$ then $B_0 \supset \mathbb{B}_{\lambda/K}$. This observation yields the following corollary.

Corollary 3.2. *For any $\lambda > 0$ and $x \in \mathbb{B}_{\lambda/K}$, $\lim_{\epsilon\to 0} \epsilon^2 \log \mathbb{E}^\epsilon_x(\tau(\mathbb{B}_\lambda)) = V_0(\mathbb{B}_\lambda)$ and for every $\alpha > 0$*

$$\lim_{\epsilon\to 0} \mathbb{P}^\epsilon_x(|\epsilon^2 \log \tau - V_0(\mathbb{B}_\lambda)| > \alpha) = 0.$$

The following lemma is from [2]. For $x \in G$, denote by $\mathcal{A}(x)$ the class of all $z \in \mathrm{AC}([0,\infty) : G)$ such that $z = \Gamma(x + \int_0^\cdot v(s)\mathrm{d}s)$ for some v satisfying $\int_0^t |v(s)|\mathrm{d}s < \infty$ and $v(t) \in \mathcal{C}(\delta)$ for all $t \ge 0$, where $\delta > 0$ is as in Condition 2.4. Let

$$T(x) = \sup_{z\in\mathcal{A}(x)} \inf\{t \ge 0 : z(t) = 0\}.$$

Lemma 3.1 ([2]). *There are positive constants $\gamma_1, \gamma_2, \gamma_3$ such that:*

(1) *For all $x, y \in G$, $|T(x) - T(y)| \le \gamma_1 \|x - y\|$.*
(2) *For all $x \in G$, $\gamma_2\|x\| \le T(x)$.*
(3) *For all $x \in G$, $t \ge 0$,*

$$T(Z(t)) \le [T(x) - t]^+ + \gamma_3 \epsilon \eta_t, \quad \mathbb{P}^\epsilon_x \text{ a.s} \tag{3.6}$$

where $\eta_t = \sup_{0\le s\le t} |\int_0^s \sigma(Z(u))\mathrm{d}W(u)|$.

We now introduce one additional condition on the reflected diffusion model. From Condition 2.3 it follows that for $t > 0$, $\mathbb{P}^\epsilon_x \circ Z(t)^{-1}$ is mutually absolutely continuous with respect to the Lebesgue measure (see the proof of Lemma 5.7 in [12]). Denote the probability density of the measure $\mathbb{P}^\epsilon_x \circ Z(t)^{-1}$ by $p_\epsilon(t, x, z)$. Let λ denote the Lebesgue measure on G.

Condition 3.1. *For every $t_1 > 0$ there is a $M_0 \equiv M_0(t_1) \in (0,\infty)$ satisfying the following. Given $M \ge M_0$, there exist finite positive constants ε_1, χ, c_1 and sets $E_\epsilon \subset \mathbb{B}_{\frac{\gamma_2}{\gamma_1}M\epsilon^2}$ such that for all $\epsilon \in (0, \varepsilon_1)$:*

$$\lambda(E_\epsilon) \ge c_1\epsilon^{2k} \quad \text{and} \quad \inf_{x\in\mathbb{B}_{M\epsilon^2}} \inf_{z\in E_\epsilon} \epsilon^{2k} p_\epsilon(t_1\epsilon^2, x, z) \ge \chi.$$

When p_ϵ is the transition density of a diffusion process in $\mathbb{R}^k$ with b, σ satisfying Conditions 2.2, 2.3, the minorization property as specified in Condition 3.1 is seen to hold using classical Aronson estimates [1] (cf. [27,14]). The following proposition gives one basic setting where the condition is satisfied for constrained diffusions. Obtaining general conditions on $(G, \mathbb{D}, b, \sigma)$ under which the above estimate holds is a challenging open problem.

Proposition 3.1. *Suppose that $b(x) \equiv b_0$, $\sigma(x) = I$ and $G = \mathbb{R}^k_+$. Also let $N = k$ and $\mathbb{D} = (I - P')$, where P is a substochastic matrix with zeros on the diagonal and spectral radius strictly less than 1. Finally suppose that the cone $\mathbb{V} = \{x \in G : \mathbb{D}'x \ge 0\}$ has a nonempty interior. Then Condition 3.1 is satisfied.*

Proof. Choose $z_0 \in G$ and $M_0, \kappa_1 > 0$ such that $\mathbb{B}_{\kappa_1}(z_0) \subset \mathbb{V} \bigcap \mathbb{B}_{\frac{\gamma_2}{\gamma_1}M_0}$ and so that $z - bt \in \mathbb{B}_{\kappa_1}(z_0)$ for all $t \le t_1$ and $z \in \mathbb{B}_{\kappa_1/2}(z_0)$. Fix $M \ge M_0$ and define

$$E_\epsilon = \{z \in G^0 : z - bt \in \mathbb{V} \text{ for all } 0 \le t \le t_1\epsilon^2\} \bigcap \mathbb{B}_{\frac{\gamma_2}{\gamma_1}M\epsilon^2}.$$

Clearly $\epsilon^2 \mathbb{B}_{\kappa_1/2}(z_0) \subset E_\epsilon$ and so for a suitable positive constant c_1, $\lambda(E_\epsilon) \ge c_1\epsilon^{2k}$.

Let, for given $x \in G$ and $t > 0$, $\mathcal{T}_\epsilon(t,x,z)$ denote the probability density of the random vector $x + b_0 t + \epsilon W_t$, where W is a standard k-dimensional Brownian motion. Then

$$\mathcal{T}_\epsilon(t,x,z) = \frac{1}{(2\pi\epsilon^2 t)^{k/2}} \mathrm{e}^{-\frac{|z-b_0t-x|^2}{2\epsilon^2 t}}.$$

The density $p_\epsilon(t,x,z)$ of $\mathbb{P}^\epsilon_x \circ Z(t)^{-1}$, for $z \in G^0$, can now be written as (see [4] and also Eq. (7.39) in [11])

$$p_\epsilon(t,x,z) = \mathcal{T}_\epsilon(t,x,z) + \lim_{n\to\infty} \mathbb{E}^\epsilon_x \int_0^{t\wedge\sigma_n} \langle \nabla \mathcal{T}_\epsilon(t-r, Z(r), z), \mathrm{d}\mathbb{D}Y(r)\rangle, \tag{3.7}$$

where $\nabla\mathcal{T}_\epsilon(t,x,z)$ denotes the gradient in variable x and $\sigma_n = \inf\{t : Z(t) \in \mathbb{B}^c_n\}$. Noting that $\nabla\mathcal{T}_\epsilon(t,x,z) = \frac{1}{t\epsilon^2}\mathcal{T}_\epsilon(t,x,z)(z-x-b_0t)$ and $\int_0^\infty Z(r)\mathrm{d}Y(r) = 0$, a.e. $\mathbb{P}^\epsilon_x$ we have that

$$\int_0^\cdot \langle \nabla\mathcal{T}_\epsilon(t-r, Z(r), z), \mathrm{d}\mathbb{D}Y(r)\rangle$$
$$\ge \int_0^\cdot \frac{1}{(t-r)\epsilon^2} \mathcal{T}_\epsilon(t-r, Z(r), z) \langle (I-P)(z - b(t-r)), \mathrm{d}Y(r)\rangle.$$

Next recall that Y is non-decreasing and since $z \in E_\epsilon$, $(I-P)(z - b(t_1\epsilon^2 - r)) \ge 0$ for all $r \le t_1\epsilon^2$. This shows that

$$\int_0^{t_1\epsilon^2} \langle \nabla\mathcal{T}_\epsilon(t_1\epsilon^2 - r, Z(r), z), \mathrm{d}\mathbb{D}Y(r)\rangle \ge 0.$$

From (3.7) we now see that, for all $x \in \mathbb{B}_{M\epsilon^2}$ and $z \in E_\epsilon$

$$p_\epsilon(t_1\epsilon^2, x, z) \ge \mathcal{T}_\epsilon(t_1\epsilon^2, x, z).$$

Using on the right side above the estimate

$$|z - x - b_0t_1\epsilon^2|^2 \le 3(|z|^2 + |x|^2 + |b_0|^2t_1^2\epsilon^4) \le 3\epsilon^4(2M^2 + |b_0|^2t_1^2),$$

we have that

$$p_\epsilon(t_1\epsilon^2, x, z) \ge \frac{\chi}{\epsilon^{2k}},$$

where $\chi = \frac{1}{(2\pi t_1)^{k/2}} \exp(\frac{-(3t_1^2b_0^2 + 6M^2)}{2t_1})$. □

We can now present the main result of this section. Recall that Conditions 2.1, 2.2 and 2.4 are assumed throughout this work. Denote by $\mathcal{H}$ the collection of all $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying $\sup_{t>0} \frac{|\psi(t)|}{(1+\log^+(t))^q} < \infty$, for some $0 < q < 1$, and $\frac{\sup_{s,t\ge 0:|t-s|\le r} |\psi(t)-\psi(s)|}{r^m+1} < \infty$ for some $m \ge 1$.

Theorem 3.2. *Suppose that Condition 3.1 holds. Let $\mathbb{K}$ be a compact subset of B_0. Then the following hold.*

(i) *For every bounded measurable map $f : \bar{B} \to \mathbb{R}$ there is a $\delta_1 \in (0,\infty)$ such that*

$$\lim_{\epsilon\to 0} \sup_{x,y\in\mathbb{K}} \mathrm{e}^{\delta_1/\epsilon} |\mathbb{E}^\epsilon_x f(Z(\tau)) - \mathbb{E}^\epsilon_y f(Z(\tau))| = 0.$$

(ii) *For every* $\psi \in \mathcal{H}$

$$\limsup_{\epsilon\to 0}\sup_{x,y\in\mathbb{K}} |\mathbb{E}^{\epsilon}_x\psi(\tau) - \mathbb{E}^{\epsilon}_y\psi(\tau)| < \infty.$$

The rest of this section is devoted to the proof of Theorem 3.2. We begin with some preliminary lemmas.

Lemma 3.2. (i) *There exist positive constants* $\upsilon_1, \upsilon_2, \upsilon_3$ *and* ε_2 *such that for all* $t > 0$ *and* $\epsilon \in (0, \varepsilon_2)$

$$\sup_{x\in\mathbb{B}_{\upsilon_1}} \mathbb{P}^{\epsilon}_x(\tau \le t) \le e^{-\upsilon_2/\epsilon} e^{\upsilon_3 t}.$$

(ii) *For each compact set* $\mathbb{K} \subset B_0$ *and* $t > 0$*, there exist positive constants* υ_4 *and* ε_3 *such that for all* $\epsilon \in (0, \varepsilon_3)$

$$\inf_{x\in\mathbb{K}} \mathbb{P}^{\epsilon}_x(\tau > t) \ge 1 - e^{-\frac{\upsilon_4}{\epsilon}}.$$

Proof. Choose κ_1 small enough so that $\bar{\mathbb{B}}_{\kappa_1} \subset B$. Using Lemma 2.1, fix $0 < \upsilon_1 < \kappa_1$ sufficiently small such that for every $x \in \bar{\mathbb{B}}_{\upsilon_1}$ and $z \in \mathcal{A}(x)$, $z(t) \in \mathbb{B}_{\kappa_1/2}$ for all $t \ge 0$. Fix $x \in \bar{\mathbb{B}}_{\upsilon_1}$ and define $\tilde{\xi}_x$ as

$$\tilde{\xi}_x(t) = \Gamma(x + \int_0^{\cdot} b(Z(s))ds)(t), \quad t \ge 0.$$

Note that, by Condition 2.4, $\tilde{\xi}_x \in \mathcal{A}(x)$ and so

$$\mathbb{P}^{\epsilon}_x(\tau \le t) \le \mathbb{P}^{\epsilon}_x\left(\sup_{0\le s\le t} |Z(s) - \tilde{\xi}_x(s)| > \frac{\kappa_1}{2}\right). \tag{3.8}$$

Thus using the Lipschitz property of the Skorokhod map

$$\mathbb{P}^{\epsilon}_x(\tau \le t) \le \mathbb{P}^{\epsilon}_x\left(K\epsilon \sup_{0\le s\le t}\left|\int_0^s \sigma(Z(u))dW(u)\right| > \kappa_1/2\right) \le \kappa_2 e^{-\frac{\kappa_1}{2K\epsilon}} e^{\kappa_3 t},$$

for suitable $\kappa_2, \kappa_3 \in (0, \infty)$. Choose ε_2 small enough so that $\kappa_2 e^{-\frac{\kappa_1}{4K\epsilon}} \le 1$ for all $\epsilon < \varepsilon_2$. Part (i) now follows on setting $\upsilon_2 = \kappa_1/4K$ and $\upsilon_3 = \kappa_3$.

We now consider Part (ii). Fix $x \in \mathbb{K}$. By definition of B_0, for some $\kappa_1 > 0$ and all $x \in \mathbb{K}$, $\text{dist}(\mathcal{S}_x, \partial B) \ge \kappa_1$, where $\mathcal{S}_x$ is as defined below (3.1). Therefore using the Lipschitz property of Γ and Gronwall's inequality we have for $x \in \mathbb{K}$

$$\sup_{0\le s\le t} |Z(s) - \xi_x(s)| \le e^{K\gamma t} K\epsilon \sup_{0\le s\le t}\left|\int_0^s \sigma(Z(u))dW(u)\right|.$$

Hence arguing as before we have for $x \in \mathbb{K}$

$$\mathbb{P}^{\epsilon}_x(\tau \le t) \le \mathbb{P}^{\epsilon}_x\left(e^{K\gamma t} K\epsilon \sup_{0\le s\le t}\left|\int_0^s \sigma(Z(u))dW(u)\right| > \kappa_1/2\right) \le \kappa_2 e^{-\frac{\kappa_1 e^{-K\gamma t}}{2K\epsilon}} e^{\kappa_3 t}.$$

The result follows. □

The following lemma is proved along the lines of Theorem 4.1 of [2].

Lemma 3.3. *There are positive constants $L_0, \upsilon_6, \upsilon_7$ and ε_4 such that for all $\epsilon \in (0, \varepsilon_4)$ and $\ell \geq L_0$*

$$\sup_{x \in \bar{B}} \mathbb{P}_x^\epsilon(\sigma^\epsilon(\ell) > t) \leq \exp\left(\frac{\upsilon_7 - \upsilon_6 t}{\epsilon^2}\right), \quad t \geq 0,$$

where $\sigma^\epsilon(\ell) = \inf\{t : |Z(t)| \leq \ell\epsilon^2\}$.

Proof. For $\ell > 0$ and $\epsilon > 0$, let $\ell^\epsilon = \ell\epsilon^2$ and $A_n^\epsilon(\ell) = \{\omega : \inf_{s \in [0, n\ell^\epsilon]} T(Z(s)) > \ell^\epsilon\}$, $n \geq 1$. Then (cf. Eq. (4.7) in [2])

$$\mathbb{P}_x^\epsilon(A_n^\epsilon) \leq \mathbb{P}_x^\epsilon\left(\ell^\epsilon < T(Z(n\ell^\epsilon)) \leq T(x) - n\ell^\epsilon + \gamma_1 \epsilon K \sum_{j=1}^n \nu_j^\epsilon\right),$$

where $\nu_j^\epsilon = \sup_{(j-1)\ell^\epsilon \leq s \leq j\ell^\epsilon} |\int_{(j-1)\ell^\epsilon}^s \sigma(Z(u)) \mathrm{d}W(u)|$ and γ_1, K are as in Lemma 3.1 and Condition 2.1, respectively. Next, for a suitable $\kappa_1 > 0$ we have, for each $\theta > 0$ (see [2], page 993, expression below Eq. (4.7)),

$$\mathbb{P}_x^\epsilon\left(\gamma_1 \epsilon K \sum_{j=1}^n \nu_j^\epsilon \geq (n+1)\ell^\epsilon - T(x)\right)$$
$$\leq \exp(\theta T(x) - \theta \ell^\epsilon) \exp((\kappa_1 \theta^2 \epsilon^4 - \theta \epsilon^2 + (\log 8)(2\ell)^{-1}) n\ell). \tag{3.9}$$

Take $\upsilon_6 = \frac{1}{8\kappa_1}$. Choose L_0 large enough so that $(\log 8)\ (2L_0)^{-1} < \upsilon_6$. Take $\theta = 4\upsilon_6/\epsilon^2$ and fix κ_2 such that $\sup_{x \in \bar{B}} T(x) \leq \kappa_2$. Then the expression on the right side of (3.9), for all $\ell \geq L_0$, is bounded by

$$\exp\left(\frac{\upsilon_6(4\kappa_2 - n\ell^\epsilon)}{\epsilon^2} - 4\upsilon_6 \ell\right).$$

Choose $\varepsilon_4 > 0$ such that $\gamma_2 L_0 \varepsilon_4^2 \leq \kappa_2$. Fix $\epsilon \in (0, \varepsilon_4)$ and $t \geq 4\gamma_2 L_0 \epsilon^2$. Then there exists an $n_0 \in \mathbb{N}$ such that $t \in [(n_0+3)\gamma_2 L_0 \epsilon^2, (n_0+4)\gamma_2 L_0 \epsilon^2]$. Setting $\upsilon_7 = 4\upsilon_6 \kappa_2$, we have,

$$\begin{aligned}
\mathbb{P}_x^\epsilon(\sigma^\epsilon(L_0) > t) &\leq \mathbb{P}_x^\epsilon\left(\min_{0 \leq s \leq n_0 L_0^\epsilon} |Z(s)| > \ell^\epsilon\right) \\
&\leq \mathbb{P}_x^\epsilon(A_{n_0}^\epsilon(\gamma_2 L_0)) \\
&\leq \exp\left(\frac{\upsilon_7 - \upsilon_6(n_0+4)\gamma_2 L_0 \epsilon^2}{\epsilon^2}\right) \\
&\leq \exp\left(\frac{\upsilon_7 - \upsilon_6 t}{\epsilon^2}\right).
\end{aligned}$$

The inequality in the above calculation is trivial for $t < 4\gamma_2 L_0 \epsilon^2$, since for such t, $\upsilon_7 - \upsilon_6 t \geq 4\upsilon_6(\kappa_2 - \gamma_2 L_0 \epsilon^2) \geq 0$. Thus the inequality holds for all $t \geq 0$. Finally the result follows on noting that $\sigma^\epsilon(\ell)$ is monotonically decreasing in ℓ. □

An important consequence of Lemmas 3.2 and 3.3 is the following.

Lemma 3.4. *For every compact $\mathbb{K} \subset B_0$ there are positive constants υ_8 and ε_5 such that for all $\epsilon \in (0, \varepsilon_5)$ and $\ell \geq L_0$, $\inf_{x \in \mathbb{K}} \mathbb{P}_x^\epsilon(\sigma^\epsilon(\ell) < \tau) \geq 1 - \mathrm{e}^{-\upsilon_8/\epsilon}$, where $\sigma^\epsilon(\ell)$ is as in Lemma 3.3.*

Proof. Fix $x \in \mathbb{K}$ and $\ell \geq L_0$. Then for every $t > 0$,

$$\mathbb{P}_x^{\epsilon}(\sigma^{\epsilon}(\ell) \geq \tau) \leq \mathbb{P}_x^{\epsilon}(\tau \leq t) + \mathbb{P}_x^{\epsilon}(\sigma^{\epsilon}(\ell) > t).$$

Let $\upsilon_6, \upsilon_7, \varepsilon_4$ be as in Lemma 3.3. Fix $t = 2\upsilon_7/\upsilon_6$. For this choice of t and $\mathbb{K}$ as in the statement of the Lemma, let $\upsilon_4, \varepsilon_3$ be as in Lemma 3.2(ii). Choosing $\varepsilon_5 = \varepsilon_3 \wedge \varepsilon_4$ we have for $\epsilon \in (0, \varepsilon_5)$

$$\mathbb{P}_x^{\epsilon}(\sigma^{\epsilon}(\ell) \geq \tau) \leq \mathrm{e}^{-\frac{\upsilon_4}{\epsilon}} + \mathrm{e}^{-\frac{\upsilon_7}{\epsilon^2}}.$$

The result follows. $\square$

The main ingredient in the proof of Theorem 3.2 is the following coupling result which will be proved in Section 4.

Theorem 3.3. *Suppose that Condition 3.1 holds. There are finite positive constants $\delta_2, \varepsilon_6, c_2$ and $M \in [L_0, \infty)$, such that for every $\epsilon \in (0, \varepsilon_6)$ and $x_\epsilon, y_\epsilon \in \mathbb{B}_{M\epsilon^2}$, the following holds. There is a filtered probability space $(\Omega^\epsilon, \mathcal{G}^\epsilon, \{\mathcal{G}_t^\epsilon\}_{t\geq 0}, \hat{\mathbb{P}}^\epsilon)$ on which are given continuous adapted $G \times G$ valued process $\hat{Z} = (\hat{Z}^1, \hat{Z}^2)$ and a stopping time ϱ_ϵ such that:*

(i) $\hat{\mathbb{E}}^\epsilon(\mathrm{e}^{\delta_2 \varrho_\epsilon/\epsilon^2}) \leq c_2$
(ii) $\hat{\mathbb{P}}^\epsilon \circ (\hat{Z}^1)^{-1} = \mathbb{P}_{x_\epsilon}^\epsilon \circ (Z)^{-1}$, $\hat{\mathbb{P}}^\epsilon \circ (\hat{Z}^2)^{-1} = \mathbb{P}_{y_\epsilon}^\epsilon \circ (Z)^{-1}$
(iii) *for every $A \in \mathcal{B}(C([0, \infty) : G))$*

$$\hat{\mathbb{P}}^\epsilon(\hat{Z}^1(\varrho_\epsilon + \cdot) \in A \mid \mathcal{G}_{\varrho_\epsilon}^\epsilon) = \hat{\mathbb{P}}^\epsilon(\hat{Z}^2(\varrho_\epsilon + \cdot) \in A \mid \mathcal{G}_{\varrho_\epsilon}^\epsilon), \quad \textit{a.e.}$$

Remark. Part (iii) of the above theorem in particular implies that for every bounded measurable map $H : G \times [0, \infty] \to \mathbb{R}$.

$$\mathbb{I}_{\varrho_\epsilon \leq \tau_1 \wedge \tau_2} \hat{\mathbb{E}}^\epsilon(H(\hat{Z}^1(\tau_1), \tau_1) \mid \mathcal{G}_{\varrho_\epsilon}^\epsilon) = \mathbb{I}_{\varrho_\epsilon \leq \tau_1 \wedge \tau_2} \hat{\mathbb{E}}^\epsilon(H(\hat{Z}^2(\tau_2), \tau_2) \mid \mathcal{G}_{\varrho_\epsilon}^\epsilon), \quad \text{a.e.}$$

where $\tau_i = \inf\{t > 0 : \hat{Z}^i(t) \in B^c\}$, $i = 1, 2$. This observation will play a key role in the proof of Theorem 3.2.

The following proposition is a key step in the proof of Theorem 3.2. Part (ii) of the proposition is a result of independent interest.

Proposition 3.2. *Suppose Condition 3.1 holds. Let $M \geq L_0$ be as in Theorem 3.3. Then*

(i) *For every bounded measurable function $f : G \to \mathbb{R}$ there exists $\delta_3 \in (0, \infty)$ such that as $\epsilon \to 0$,*

$$\mathrm{e}^{\delta_3/\epsilon} \sup_{x,y \in \mathbb{B}_{M\epsilon^2}} |\mathbb{E}_x^\epsilon f(Z(\tau)) - \mathbb{E}_y^\epsilon f(Z(\tau))| \to 0.$$

(ii) *For every measurable function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ that satisfies $\sup_{t>0} \frac{|\psi(t)|}{(1+\log^+(t))^q} < \infty$, for some $0 < q < 1$, there exists $\delta_4 \in (0, \infty)$ such that, as $\epsilon \to 0$,*

$$\mathrm{e}^{\delta_4/\epsilon} \sup_{x,y \in \mathbb{B}_{M\epsilon^2}} |\mathbb{E}_x^\epsilon \psi(\tau) - \mathbb{E}_y^\epsilon \psi(\tau)| \to 0.$$

Proof. Let $\epsilon \in (0, \varepsilon_6)$, where ε_6 is as in Theorem 3.3. Fix $x_\epsilon, y_\epsilon \in \mathbb{B}_{M\epsilon^2}$ and let $(\Omega^\epsilon, \mathcal{G}^\epsilon, \{\mathcal{G}_t^\epsilon\}_{t\geq 0}, \hat{\mathbb{P}}^\epsilon)$, $\hat{Z} = (\hat{Z}^1, \hat{Z}^2)$ and ϱ_ϵ be as in Theorem 3.3. Let f be as in the statement of the theorem. Then

$$\mathbb{E}_{x_\epsilon}^\epsilon(f(Z(\tau))) = \hat{\mathbb{E}}^\epsilon(f(\hat{Z}^1(\tau_1))) = \hat{\mathbb{E}}^\epsilon(\mathbb{I}_{\varrho_\epsilon \leq \tau_1 \wedge \tau_2} f(\hat{Z}^1(\tau_1))) + \hat{\mathbb{E}}^\epsilon(\mathbb{I}_{\varrho_\epsilon > \tau_1 \wedge \tau_2} f(\hat{Z}^1(\tau_1))).$$

In view of the remark below Theorem 3.3

$$\begin{aligned}\hat{\mathbb{E}}^\epsilon(\mathbb{I}_{\varrho_\epsilon\le\tau_1\wedge\tau_2} f(\hat{Z}^1(\tau_1))) &= \hat{\mathbb{E}}^\epsilon(\mathbb{I}_{\varrho_\epsilon\le\tau_1\wedge\tau_2} f(\hat{Z}^2(\tau_2)))\\ &= \hat{\mathbb{E}}^\epsilon(f(\hat{Z}^2(\tau_2))) - \hat{\mathbb{E}}^\epsilon(\mathbb{I}_{\varrho_\epsilon>\tau_1\wedge\tau_2} f(\hat{Z}^2(\tau_2))).\end{aligned}$$

Also, for every $t > 0$

$$\begin{aligned}\sum_{i=1,2}\hat{\mathbb{E}}^\epsilon(\mathbb{I}_{\varrho_\epsilon>\tau_1\wedge\tau_2} f(\hat{Z}^i(\tau_i))) &\le 2|f|_\infty \sum_{i=1,2}\hat{\mathbb{P}}^\epsilon(\varrho_\epsilon > \tau_i)\\ &\le 2|f|_\infty(\mathbb{P}^\epsilon_{x_\epsilon}(\tau\le t) + \mathbb{P}^\epsilon_{y_\epsilon}(\tau\le t) + 2\hat{\mathbb{P}}^\epsilon(\varrho_\epsilon\ge t)).\end{aligned}$$

Choose ϵ_0 sufficiently small such that $M\epsilon^2 \le \upsilon_1$ for all $\epsilon \le \epsilon_0$, where υ_1 is as in Lemma 3.2 (i). Then for all $\epsilon \in (0, \epsilon_0 \wedge \varepsilon_2 \wedge \varepsilon_6)$

$$\mathbb{P}^\epsilon_{x_\epsilon}(\tau\le t) + \mathbb{P}^\epsilon_{y_\epsilon}(\tau\le t) \le 2\mathrm{e}^{-\upsilon_2/\epsilon}\mathrm{e}^{\upsilon_3 t}.$$

Next, from Theorem 3.3, $\hat{\mathbb{P}}^\epsilon(\varrho_\epsilon \ge t) \le c_2\mathrm{e}^{-\delta_2 t/\epsilon^2}$. Since $x_\epsilon, y_\epsilon \in \mathbb{B}_{M\epsilon^2}$ are arbitrary, we have

$$\sup_{x,y\in\mathbb{B}_{M\epsilon^2}} (\mathbb{P}^\epsilon_{x_\epsilon}(\tau\le t) + \mathbb{P}^\epsilon_{y_\epsilon}(\tau\le t) + 2\hat{\mathbb{P}}^\epsilon(\varrho_\epsilon\ge t)) \le 2\mathrm{e}^{-\upsilon_2/\epsilon}\mathrm{e}^{\upsilon_3 t} + 2c_2\mathrm{e}^{-\delta_2 t/\epsilon^2}.$$

Combining the above estimates we now have

$$\sup_{x,y\in\mathbb{B}_{M\epsilon^2}} |\mathbb{E}^\epsilon_x f(Z(\tau)) - \mathbb{E}^\epsilon_y f(Z(\tau))| \le 4|f|_\infty(\mathrm{e}^{-\upsilon_2/\epsilon}\mathrm{e}^{\upsilon_3 t} + c_2\mathrm{e}^{-\delta_2 t/\epsilon^2}).$$

Part (i) of the proposition now follows on sending $\epsilon \to 0$.

Now consider part (ii). An argument similar to above yields, for all $\epsilon \in (0, \varepsilon_6)$ and x_ϵ, $y_\epsilon \in \mathbb{B}_{M\epsilon^2}$

$$|\mathbb{E}^\epsilon_{x_\epsilon}\psi(\tau) - \mathbb{E}^\epsilon_{y_\epsilon}\psi(\tau)| \le \beta_1(\epsilon) + \beta_2(\epsilon),$$

where $\beta_i(\epsilon) = \hat{\mathbb{E}}^\epsilon(\mathbb{I}_{\varrho_\epsilon>\tau_1\wedge\tau_2}\psi(\tau_i))$, $i = 1, 2$. We now use the logarithmic growth condition on ψ. From Theorem 3.1(i) and Jensen's inequality, we can find $\epsilon_1 \in (0, \varepsilon_6)$ and $\kappa_1 > 0$ such that for all $\epsilon < \epsilon_1$, $\hat{\mathbb{E}}^\epsilon(|\psi(\tau_i)|^{1/q}) \le \frac{\kappa_1}{\epsilon^2}$. Thus for all $\epsilon \in (0, \epsilon_0 \wedge \epsilon_1 \wedge \varepsilon_2)$

$$\beta_i(\epsilon) \le (\hat{\mathbb{P}}^\epsilon(\varrho_\epsilon > \tau_1\wedge\tau_2))^{1-q}(\hat{\mathbb{E}}^\epsilon\psi^{\frac{1}{q}}(\tau_i))^q \le (2\mathrm{e}^{-\upsilon_2/\epsilon}\mathrm{e}^{\upsilon_3 t} + 2c_2\mathrm{e}^{-\delta_2 t/\epsilon^2})^{(1-q)}\left(\frac{\kappa_1}{\epsilon^2}\right)^q.$$

Part (ii) of the proposition now follows on sending $\epsilon \to 0$ in the above equation. □

Proof of Theorem 3.2. Fix $M \ge L_0$ as in the statement of Theorem 3.3. We first consider part (i). Fix a compact set $\mathbb{K} \subset B_0$ and a bounded measurable map $f : G \to \mathbb{R}$. For each $\epsilon > 0$, fix $x_\epsilon \in \mathbb{B}_{M\epsilon^2}$. It suffices to show that, for some $\delta_1 \in (0, \infty)$,

$$\mathrm{e}^{\delta_1/\epsilon}\sup_{x\in\mathbb{K}}|\mathbb{E}^\epsilon_x f(Z(\tau)) - \mathbb{E}^\epsilon_{x_\epsilon} f(Z(\tau))| \to 0, \quad \text{as } \epsilon \to 0. \tag{3.10}$$

Write $\sigma^\epsilon(M) = \sigma^\epsilon$, where $\sigma^\epsilon(\cdot)$ is as in Lemma 3.3. Note that

$$\mathbb{E}^\epsilon_x f(Z(\tau)) = \mathbb{E}^\epsilon_x(\mathbb{I}_{\sigma^\epsilon\le\tau} f(Z(\tau))) + \mathbb{E}^\epsilon_x(\mathbb{I}_{\sigma^\epsilon>\tau} f(Z(\tau))).$$

Thus

$$\begin{aligned}|\mathbb{E}^\epsilon_x(f(Z(\tau))) - \mathbb{E}^\epsilon_{x_\epsilon} f(Z(\tau))| \le{}& \mathbb{E}^\epsilon_x(\mathbb{I}_{\sigma^\epsilon\le\tau}|\mathbb{E}^\epsilon_{Z(\sigma^\epsilon)} f(Z(\tau)) - \mathbb{E}^\epsilon_{x_\epsilon} f(Z(\tau))|)\\ &+ 2|f|_\infty\mathbb{P}^\epsilon_x(\sigma^\epsilon > \tau).\end{aligned}$$

From Proposition 3.2, we can find $\epsilon_1 \in (0, \infty)$ such that for all $\epsilon \in (0, \epsilon_1)$

$$\mathbb{I}_{\sigma^\epsilon \le \tau} |\mathbb{E}^\epsilon_{Z(\sigma^\epsilon)} f(Z(\tau)) - \mathbb{E}^\epsilon_{x_\epsilon} f(Z(\tau))| \le \mathrm{e}^{-\delta_3/\epsilon}, \quad \mathbb{P}^\epsilon_x \text{ a.e.}$$

Also from Lemma 3.4, we can find $\epsilon_2 \in (0, \epsilon_1)$ such that for all $\epsilon \in (0, \epsilon_2)$

$$\sup_{x \in \mathbb{K}} \mathbb{P}^\epsilon_x (\sigma^\epsilon > \tau) \le \mathrm{e}^{-\upsilon_8/\epsilon}.$$

Part (i) follows on combining the above two estimates.

We now consider part (ii). Fix ψ as in the statement of the theorem and let $x \in \mathbb{K}$. Then, on the set $\sigma^\epsilon \le \tau$,

$$\mathbb{E}^\epsilon_x (\psi(\tau) \mid \mathcal{F}_{\sigma^\epsilon}) = \mathbb{E}^\epsilon_x (\psi(\tau) - \psi(\tau - \sigma^\epsilon) \mid \mathcal{F}_{\sigma^\epsilon}) + \mathbb{E}^\epsilon_x (\psi(\tau - \sigma^\epsilon) \mid \mathcal{F}_{\sigma^\epsilon}).$$

From Proposition 3.2

$$\begin{aligned} |\mathbb{E}^\epsilon_x (\mathbb{I}_{\sigma^\epsilon \le \tau} (\mathbb{E}^\epsilon_x (\psi(\tau - \sigma^\epsilon) \mid \mathcal{F}_{\sigma^\epsilon}) - \mathbb{E}^\epsilon_{x^\epsilon} (\psi(\tau))))| &\le \sup_{x, y \in \mathbb{B}_{M\epsilon^2}} |\mathbb{E}^\epsilon_x (\psi(\tau)) - \mathbb{E}^\epsilon_y (\psi(\tau))| \\ &\to 0, \quad \text{as } \epsilon \to 0. \end{aligned}$$

Also by an application of Theorem 3.1(i), Jensen's inequality and Lemma 3.4, we can find positive constants ϵ_0 and κ_1 such that for all $\epsilon \in (0, \epsilon_0)$ and $x \in \mathbb{K}$

$$|\mathbb{E}^\epsilon_x (\mathbb{I}_{\sigma^\epsilon > \tau} \psi(\tau))| + \mathbb{P}^\epsilon_x (\sigma^\epsilon > \tau) |\mathbb{E}^\epsilon_{x_\epsilon} (\psi(\tau))| \le \frac{\kappa_1}{\epsilon^{2q}} \mathrm{e}^{-\upsilon_8(1-q)/\epsilon}.$$

The last expression approaches 0 as $\epsilon \to 0$. Finally since $\psi \in \mathcal{H}$, we have for some $m \ge 1$, positive constants κ_2, κ_3 and ϵ sufficiently small

$$\mathbb{E}^\epsilon_x (\mathbb{I}_{\sigma^\epsilon \le \tau} \mathbb{E}^\epsilon_x ((\psi(\tau) - \psi(\tau - \sigma^\epsilon)) \mid \mathcal{F}_{\sigma^\epsilon})) \le \kappa_2 (1 + \mathbb{E}^\epsilon_x (\sigma^\epsilon)^m) \le \kappa_3 (1 + \epsilon^{2m}),$$

where the last inequality follows on using Lemma 3.3. The result follows. □

4. Proof of Theorem 3.3

In this section we prove Theorem 3.3. Since the proof is quite long, we first give a brief sketch for the reader's convenience. The proof is split into various lemmas leading finally to the construction of a Markov process meeting all the requirements of Theorem 3.3. This construction proceeds by first obtaining certain discrete time processes $Z^{\epsilon,*}_n = (Z^{\epsilon,*,1}_n, Z^{\epsilon,*,2}_n)$, $n \in \mathbb{N}_0$, with analogous properties. The argument that yields the desired continuous time processes from their discrete analogs is based on a careful "surgery" that involves construction, on a certain function space, of measures by patching together of suitable conditional laws. This argument appears towards the end of the section. Construction of the discrete time sequence $(Z^{\epsilon,*}_n)$, given below (4.7), crucially uses Condition 3.1 (see Lemma 4.3 and also the minorization property (4.5)). The fact that the constructed chain satisfies properties analogous to (i)–(iii) of Theorem 3.3 can be seen through Lemma 4.4 (for (ii)), Eq. (4.13) (for (iii)) and Lemma 4.7 (for (i)). Lemma 4.7, that gives uniform in ϵ bounds on certain exponential moments of hitting times of "pseudo-atoms", is proved using Lyapunov function constructions and Foster type drift inequalities obtained in Lemmas 4.1, 4.2 and 4.5. An intermediate step in obtaining uniform estimates on hitting times of pseudo-atoms is Lemma 4.6, where similar estimates for hitting times of certain closely related sets are studied. This lemma is a key ingredient in the proof of Lemma 4.7.

Uniform Foster inequality for embedded chains. Choose $L, \alpha \in (0, \infty)$ such that

$$\alpha - \log 2 - \varsigma_1 \alpha^2 L^{-1} \equiv \beta_0 \in (0, \infty), \tag{4.1}$$

where $\varsigma_1 = k^2\gamma^2\gamma_3^2$ (γ comes from Condition 2.2). For $\epsilon > 0$ and $r \geq 1$, let $\Delta_\epsilon = L\epsilon^2$, $D_\epsilon^r = \{x \in G : T(x) \leq \Delta_\epsilon r\}$ and $U^\epsilon(x) = \exp\{\alpha\Delta_\epsilon^{-1}T(x)\}$, $x \in G$. Define for $n \in \mathbb{N}_0$, $Z_n^\epsilon = Z(n\Delta_\epsilon)$.

Lemma 4.1. *There exist $m_1 \in (0,\infty)$, $\beta \in (0,1)$ such that for all $\epsilon > 0$ and $r \geq 1$,*

$$\mathbb{E}_x^\epsilon(U^\epsilon(Z_1^\epsilon)) \leq (1-\beta)U^\epsilon(x) + m_1\mathbb{I}_{D_\epsilon^r}(x), \quad x \in G$$

and

$$\mathbb{E}_x^\epsilon(U^\epsilon(Z_1^\epsilon)) \leq (1-\beta)U^\epsilon(x)\mathbb{I}_{(D_\epsilon^1)^c}(x) + m_1\mathbb{I}_{D_\epsilon^1}(x), \quad x \in G.$$

Proof. From (3.6) we have, for all $\epsilon > 0$ and $x \in G$,

$$(U^\epsilon)^{-1}(x)\mathbb{E}_x^\epsilon(U^\epsilon(Z_1^\epsilon)) \leq \mathbb{E}_x^\epsilon \exp(\alpha\Delta_\epsilon^{-1}((T(x)-\Delta_\epsilon)^+ + \gamma_3\epsilon\eta_{\Delta_\epsilon} - T(x))).$$

For $x \in (D_\epsilon^1)^c$, the right side above equals

$$\mathbb{E}_x^\epsilon \exp(\alpha\Delta_\epsilon^{-1}(\gamma_3\epsilon\eta_{\Delta_\epsilon} - \Delta_\epsilon)) \leq 2\mathrm{e}^{-\alpha}\exp(\varsigma_1\alpha^2L^{-1}) = \mathrm{e}^{-\beta_0}.$$

Also for $x \in D_\epsilon^1$, using (3.6) once more,

$$\mathbb{E}_x^\epsilon(U^\epsilon(Z_1^\epsilon)) = \mathbb{E}_x^\epsilon \exp(\alpha\Delta_\epsilon^{-1}T(Z_1^\epsilon)) \leq \mathbb{E}_x^\epsilon \exp(\alpha\Delta_\epsilon^{-1}(\gamma_3\epsilon\eta_{\Delta_\epsilon})) \leq 2\exp(\varsigma\alpha^2L^{-1}).$$

The result now follows on setting $\beta = (1-\mathrm{e}^{-\beta_0})$ and $m_1 = 2\exp(\varsigma\alpha^2L^{-1})$. □

Pair of embedded chains for different initial conditions. Choose a filtered probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \{\tilde{\mathcal{F}}_t\})$ on which are given $\{\tilde{\mathcal{F}}_t\}$ adapted k-dimensional processes Z^i, Y^i, W^i, $i = 1,2$ and collection of probability measures $\{\tilde{\mathbb{P}}_{x,y}^\epsilon\}_{x,y\in G}$ such that, for all $x_1, x_2 \in G$, under $\tilde{\mathbb{P}}_{x_1,x_2}^\epsilon$, W^1, W^2 are independent $\{\tilde{\mathcal{F}}_t\}$-standard Brownian motions and for $i = 1,2$, (2.5) holds with (x, Z, W, Y) there replaced by (x^i, Z^i, W^i, Y^i).

Recall the parameter L_0 introduced in Lemma 3.3. Fix $r_0 \in [1,\infty)$ such that

$$\beta\exp(\alpha r_0) \geq 2m_1 \quad \text{and} \quad r_0 \geq \max(\gamma_2 M_0(L)L^{-1}, \gamma_1 L_0 L^{-1}) \tag{4.2}$$

where $M_0(\cdot)$ is as in Condition 3.1. Let $F_\epsilon^1 = D_\epsilon^{r_0}$ and $\bar{F}_\epsilon = F_\epsilon^1 \times F_\epsilon^1$. Let $H = G \times G$. Set $m_2 = 2(m_1 + \exp(\alpha r_0))$. Let $\bar{Z} = (Z^1, Z^2)$ and define for $x, y \in G$, $\bar{U}^\epsilon(x,y) = U^\epsilon(x) + U^\epsilon(y)$.

Lemma 4.2. *For all $\bar{x} = (x_1, x_2) \in H$ and $\epsilon > 0$.*

$$\mathbb{E}_{x_1,x_2}^\epsilon \bar{U}^\epsilon(\bar{Z}(\Delta_\epsilon)) \leq (1-\beta/2)\bar{U}^\epsilon(\bar{x})\mathbb{I}_{\bar{F}_\epsilon^c}(\bar{x}) + m_2\mathbb{I}_{\bar{F}_\epsilon}(\bar{x}). \tag{4.3}$$

Proof. Suppose $x_1 \in (F_\epsilon^1)^c$ and $x_2 \in (D_\epsilon^1)^c$. Then from the second expression in Lemma 4.1 we have

$$\mathbb{E}_{x_1,x_2}^\epsilon \bar{U}^\epsilon(\bar{Z}(\Delta_\epsilon)) \leq (1-\beta)\bar{U}^\epsilon(\bar{x}) \leq (1-\beta/2)\bar{U}^\epsilon(\bar{x}).$$

If $x_1 \in (F_\epsilon^1)^c$ and $x_2 \in D_\epsilon^1$, then once again from Lemma 4.1

$$\begin{aligned}\mathbb{E}_{x_1,x_2}^\epsilon \bar{U}^\epsilon(\bar{Z}(\Delta_\epsilon)) &\leq (1-\beta/2)U^\epsilon(x_1) - \beta/2U^\epsilon(x_1) + m_1 \\ &\leq (1-\beta/2)U^\epsilon(x_1) \leq (1-\beta/2)\bar{U}^\epsilon(\bar{x}),\end{aligned}$$

where the second inequality is a consequence of (4.2). This proves (4.3) for all $x_1 \in (F^1_\epsilon)^c$ and $x_2 \in G$. Similarly (4.3) is seen to hold whenever $x_2 \in (F^1_\epsilon)^c$ and $x_1 \in G$. Finally, if $\bar{x} \in \bar{F}_\epsilon$, we have from Lemma 4.1

$$\mathbb{E}^\epsilon_{x_1,x_2} \bar{U}^\epsilon(\bar{Z}(\Delta_\epsilon)) \leq 2\left(m_1 + \sup_{x \in F^1_\epsilon} U^\epsilon(x)\right) \leq 2(m_1 + \exp(\alpha r_0)) = m_2.$$

The result follows. □

Fix $M_1 = r_0 L \gamma_2^{-1}$. By our choice of r_0 (see (4.2)), $M_1 \geq M_0(L)$. Recall that $F^1_\epsilon = \{x \in G : T(x) \leq L r_0 \epsilon^2\}$. From Lemma 3.1(2), $T(x) \leq L r_0 \epsilon^2$ implies that $|x| \leq L r_0 \gamma_2^{-1} \epsilon^2 = M_1 \epsilon^2$. Also, from Lemma 3.1(i), $|x| \leq \gamma_2 \gamma_1^{-1} M_1 \epsilon^2 = L r_0 \gamma_1^{-1} \epsilon^2$ implies that $T(x) \leq L r_0 \epsilon^2$. Thus

$$\mathbb{B}_{\frac{\gamma_2}{\gamma_1} M_1 \epsilon^2} \subset F^1_\epsilon \subset \mathbb{B}_{M_1 \epsilon^2}. \tag{4.4}$$

The following lemma is now immediate from Condition 3.1.

Lemma 4.3. *There are $\theta_0 \in (0, 1)$, $m_3, \chi_1 \in (0, \infty)$, and for each $\epsilon \in (0, \theta_0)$, $F^0_\epsilon \subset F^1_\epsilon$ such that $\lambda(F^0_\epsilon) \geq m_3 \epsilon^{2k}$ and $\epsilon^{2k} p_\epsilon(\Delta_\epsilon, x, z) \geq \chi_1$ for all $x \in F^1_\epsilon$ and $z \in F^0_\epsilon$.*

Define a transition probability kernel $\bar{p}_\epsilon : H \times \mathcal{B}(H) \to [0, 1]$ as

$$\bar{p}_\epsilon(\bar{x}, A_1 \times A_2) = \int_{A_1} p_\epsilon(\Delta_\epsilon, x_1, y) \mathrm{d}y \int_{A_2} p_\epsilon(\Delta_\epsilon, x_2, y) \mathrm{d}y, \quad \bar{x} = (x_1, x_2) \in H,$$
$$A_i \in \mathcal{B}(G),\ i = 1, 2.$$

Let $\bar{F}^0_\epsilon = F^0_\epsilon \times F^0_\epsilon$ and define $\Psi_\epsilon \in \mathcal{P}(H)$ as

$$\Psi_\epsilon(A) = \frac{(\lambda \otimes \lambda)(A \cap \bar{F}^0_\epsilon)}{(\lambda(F^0_\epsilon))^2}, \quad A \in \mathcal{B}(H).$$

Let $\rho = \frac{(\chi_1^2 m_3^2 \wedge 1)}{2}$. From Lemma 4.3 we have that

$$\bar{p}_\epsilon(\bar{x}, A) \geq 2\rho \mathbb{I}_{\bar{F}_\epsilon}(\bar{x}) \Psi_\epsilon(A), \quad \bar{x} \in H,\ A \in \mathcal{B}(H). \tag{4.5}$$

Construction of the split chain. We now construct a Markov chain (referred to as the split chain) on an augmentation H^* of the space H. For $A \in \mathcal{B}(H)$, let $A^* = A \times \{0, 1\}$, $A(0) = A \times \{0\}$ and $A(1) = A \times \{1\}$. We will denote by $\mathcal{B}(H^*)$ the σ-field on H^* generated by $\{A(0), A(1) : A \in \mathcal{B}(H)\}$. For every $\mu \in \mathcal{P}(H)$ we define a $\mu^* \in \mathcal{P}(H^*)$ as follows. For $A \in \mathcal{B}(H)$:

$$\mu^*(A(0)) = (1 - \rho)\mu(A \cap \bar{F}_\epsilon) + \mu(A \cap \bar{F}^c_\epsilon), \qquad \mu^*(A(1)) = \rho\mu(A \cap \bar{F}_\epsilon). \tag{4.6}$$

Clearly, $\mu^*(A(0)) + \mu^*(A(1)) = \mu(A)$ and if $A \subset (\bar{F}_\epsilon)^c$, then $\mu^*(A(0)) = \mu(A)$.

For $\epsilon \in (0, \theta_0)$, we define an H^* valued Markov chain $X^{\epsilon,*}_n \equiv (Z^{\epsilon,*}_n, i^*_n)$, $n \in \mathbb{N}_0$ where $Z^{\epsilon,*}_n \equiv (Z^{\epsilon,*,1}_n, Z^{\epsilon,*,2}_n)$, with transition probability kernel $q^\epsilon : H^* \times \mathcal{B}(H^*)$ defined as follows. For $z^* \equiv (\bar{z}, i) \in H^*$ and $E \in \mathcal{B}(H^*)$

$$q^\epsilon(z^*, E) = \begin{cases} \bar{p}^*_\epsilon(\bar{z}, E) & \text{if } z^* \in H(0) \setminus \bar{F}_\epsilon(0), \\ \dfrac{1}{1 - \rho}(\bar{p}^*_\epsilon(\bar{z}, E) - \rho\Psi^*_\epsilon(E)) & \text{if } z^* \in \bar{F}_\epsilon(0), \\ \Psi^*_\epsilon(E) & \text{if } z^* \in H(1). \end{cases} \tag{4.7}$$

Let the probability law on $((H^*)^\infty, \mathcal{B}(H^*)^{\otimes\infty})$ of the sequence $\{X_n^{\epsilon,*}\}_{n\in\mathbb{N}_0}$ with $X_0^{\epsilon,*} \equiv x^* \in H^*$ be denoted by $\mathbb{Q}_{x^*}^{\epsilon,*}$. The corresponding expectation operator is denoted by $\mathbb{E}_{x^*}^*$. If $X_0^{\epsilon,*}$ has probability law $\mu^* \in \mathcal{P}(H^*)$, we will denote the probability law of $\{X_n^{\epsilon,*}\}_{n\in\mathbb{N}_0}$ by $\mathbb{Q}_{\mu^*}^{\epsilon,*}$ and the corresponding expectation operator by $\mathbb{E}_{\mu^*}^*$. For $\bar{x} \in H$, define $\mu_{\bar{x}} \in \mathcal{P}(H^*)$ as:

$$\mu_{\bar{x}}(A(0)) = (1-\rho)\mathbb{I}_{A\cap\bar{F}_\epsilon}(\bar{x}) + \mathbb{I}_{A\cap\bar{F}_\epsilon^c}(\bar{x}), \qquad \mu_{\bar{x}}(A(1)) = \rho\mathbb{I}_{A\cap\bar{F}_\epsilon}(\bar{x}), \quad A \in \mathcal{B}(H).$$

Denote by $\chi \equiv (\chi_n)_{n\in\mathbb{N}_0} = (z_n, i_n)_{n\in\mathbb{N}_0}$, where $z_n = (z_n^1, z_n^2)$, the canonical sequence on $((H^*)^\infty, \mathcal{B}(H^*)^{\otimes\infty})$. Let $\mathcal{G}_n = \sigma\{\chi_m : m \le n\}$. Denote the probability law induced by $\{\bar{Z}(n\Delta_\epsilon)\}_{n\in\mathbb{N}_0}$ on $(H^\infty), \mathcal{B}(H^{\otimes\infty})$, under $\tilde{\mathbb{P}}_{\bar{x}}^\epsilon$, by $\mathbb{Q}_{\bar{x}}^\epsilon$. The following lemma is immediate from the above construction (see [24]).

Lemma 4.4. *For all $\bar{x} \in H$ and $\epsilon \in (0, \theta_0)$, $\mathbb{Q}_{\mu_{\bar{x}}}^{\epsilon,*} \circ (z)^{-1} = \mathbb{Q}_{\bar{x}}^\epsilon$.*

Hitting time for pseudo-atom: exponential moment estimates. We now derive estimates on the hitting time of the set $\bar{F}_\epsilon(1)$. This set is referred to as a pseudo-atom for the split chain for reasons discussed below the proof of Lemma 4.7. Define, for $\epsilon > 0$, $\mathcal{V}_\epsilon : H^* \to R_+$ as $\mathcal{V}_\epsilon(x_1, x_2, i) = \bar{U}_\epsilon(x_1, x_2)$, where $(x_1, x_2, i) \in H^*$. It is easy to check from Lemma 4.2 that for all $\bar{z} \in H$ and $\epsilon \in (0, \theta_0)$

$$\mathbb{E}_{(\bar{z},0)}^*(\mathcal{V}_\epsilon(\chi_1)) - \mathcal{V}_\epsilon((\bar{z},0)) \le -\frac{\beta}{2}\mathcal{V}_\epsilon((\bar{z},0))\mathbb{I}_{(\bar{F}_\epsilon)^c}(\bar{z}) + m_4\mathbb{I}_{\bar{F}_\epsilon}(\bar{z}),$$

where $m_4 = m_2/(1-\rho)$. Also, for all $\bar{z} \in \bar{F}_\epsilon$

$$\mathbb{E}_{(\bar{z},1)}^*(\mathcal{V}_\epsilon(\chi_1)) - \mathcal{V}_\epsilon((\bar{z},1)) \le 2e^{\alpha r_0} \le m_4.$$

Combining the above estimates we have the following lemma.

Lemma 4.5. *For all $z^* \in H^*$ and $\epsilon \in (0, \theta_0)$*

$$\mathbb{E}_{z^*}^*\left(\mathcal{V}_\epsilon(\chi_{n+1}) \mid \mathcal{G}_n\right) - \mathcal{V}_\epsilon(\chi_n) \le -\frac{\beta}{2}\mathcal{V}_\epsilon(\chi_n)\mathbb{I}_{(\bar{F}_\epsilon^*)^c}(\chi_n) + m_4\mathbb{I}_{\bar{F}_\epsilon^*}(\chi_n),$$

a.e. $\mathbb{Q}_{z^*}^{\epsilon,*}$ on $\{\chi_n \in H^* \setminus (\bar{F}_\epsilon)^c(1)\}$.

Let $\bar{\varrho}_\epsilon = \min\{n \ge 1 : z_n \in \bar{F}_\epsilon\}$.

Lemma 4.6. *There are $m_5, \delta_4 \in (0, \infty)$ such that for all $\epsilon \in (0, \theta_0)$, $\sup_{x^*\in\bar{F}_\epsilon^*} \mathbb{E}_{x^*}^*(e^{\delta_4\bar{\varrho}_\epsilon}) \le m_5$.*

Proof. Fix $x^* \in \bar{F}_\epsilon^*$. Note that $\mathbb{Q}_{x^*}^{\epsilon,*}\{\chi_n \in H^* \setminus (\bar{F}_\epsilon)^c(1)\} = 1$ for all $n \ge 1$. Thus from Lemma 4.5, for $n \ge 0$,

$$\mathbb{E}_{x^*}^*(\mathcal{V}_\epsilon(\chi_{n+1}) \mid \mathcal{G}_n) \le \kappa^{-1}\mathcal{V}_\epsilon(\chi_n) - \frac{\beta}{4}\mathcal{V}_\epsilon(\chi_n) + \tilde{m}_5\mathbb{I}_{\bar{F}_\epsilon^*}(\chi_n),$$

where $\kappa = \frac{1}{(1-\beta/4)}$ and $\tilde{m}_5 = m_4 + 2e^{\alpha r_0}$. Define for $n \ge 0$, $\Phi_n^\epsilon = \kappa^n\mathcal{V}_\epsilon(\chi_n)$. Then

$$\begin{aligned}\mathbb{E}_{x^*}^*(\Phi_{n+1}^\epsilon \mid \mathcal{G}_n) &= \kappa^{n+1}\mathbb{E}_{x^*}^*(\mathcal{V}_\epsilon(\chi_{n+1}) \mid \mathcal{G}_n)\\ &\le \kappa^{n+1}\left(\kappa^{-1}\mathcal{V}_\epsilon(\chi_n) - \frac{\beta}{4}\mathcal{V}_\epsilon(\chi_n) + \tilde{m}_5\mathbb{I}_{\bar{F}_\epsilon^*}(\chi_n)\right)\\ &= \Phi_n^\epsilon - \frac{\beta}{4}\kappa^{n+1}\mathcal{V}_\epsilon(\chi_n) + \tilde{m}_5\kappa^{n+1}\mathbb{I}_{\bar{F}_\epsilon^*}(\chi_n).\end{aligned} \tag{4.8}$$

Note that, for all $N \in \mathbb{N}_0$

$$\mathbb{E}^*_{x^*}\left(\sum_{m=1}^{\bar{\varrho}_\epsilon \wedge N}(\Phi^\epsilon_m - \mathbb{E}^*_{x^*}(\Phi^\epsilon_m \mid \mathcal{G}_{m-1}))\right) = 0.$$

From (4.8), we have that on the set $\{1 < m \le \bar{\varrho}_\epsilon\}$,

$$\frac{\beta}{4}\kappa^m \mathcal{V}_\epsilon(\chi_{m-1}) \le -\mathbb{E}^*_{x^*}(\Phi^\epsilon_m \mid \mathcal{G}_{m-1}) + \Phi^\epsilon_{m-1}, \quad \mathbb{Q}^{\epsilon,*}_{x^*}\text{a.e.}$$

Combining the above two equations we have

$$\frac{\beta}{4}\mathbb{E}^*_{x^*}\left(\sum_{m=1}^{\bar{\varrho}_\epsilon \wedge N}\kappa^m \mathcal{V}_\epsilon(\chi_{m-1})\right) \le \mathcal{V}_\epsilon(x^*) + \tilde{m}_5.$$

Since $\mathcal{V}_\epsilon \ge 2$, we have

$$\mathbb{E}^*_{x^*}(\kappa^{\bar{\varrho}_\epsilon \wedge N}) \le \frac{2(\kappa-1)}{\beta\kappa}(\mathcal{V}_\epsilon(x^*) + \tilde{m}_5) + 1.$$

The result follows on sending $N \to \infty$. $\square$

Let $\varrho^*_\epsilon = \inf\{n \in \mathbb{N}_0 : \chi_n \in (\bar{F}_\epsilon)(1)\}$.

Lemma 4.7. *There are $m_6, \delta_5 \in (0,\infty)$ such that for $\epsilon \in (0,\theta_0)$, $\sup_{x^* \in (\bar{F}_\epsilon)^*}\mathbb{E}^*_{x^*}(e^{\delta_5 \varrho^*_\epsilon}) \le m_6$.*

Proof. Note that if $x^* \in (\bar{F}_\epsilon)(1)$, $\varrho^*_\epsilon = 0$. Consider now $x^* \in (\bar{F}_\epsilon)(0)$. From Lemma 4.6, for all $\epsilon \in (0,\theta_0)$, $\sup_{x^* \in \bar{F}^*_\epsilon}\mathbb{E}^*_{x^*}(e^{\delta_4 \bar{\varrho}_\epsilon}) \le m_5$. For $\gamma \in (0,\delta_4]$ define $L(\gamma) = \sup_{x^* \in \bar{F}^*_\epsilon}\mathbb{E}^*_{x^*}(e^{\gamma \bar{\varrho}_\epsilon})$. Then

$$L(\gamma)(1-\rho) \le (L(\delta_4))^{\gamma/\delta_4}(1-\rho) \le (m_5)^{\gamma/\delta_4}(1-\rho).$$

Since $(1-\rho) < 1$, we can find $\delta_5 \in (0,\delta_4)$ such that

$$\bar{L} \equiv L(\delta_5) < (1-\rho)^{-1}. \tag{4.9}$$

Define a sequence of stopping times $\{\varpi^\epsilon_n\}$ as $\varpi^\epsilon_0 = 0$ and $\varpi^\epsilon_m = \inf\{n > \varpi^\epsilon_{m-1} : \chi_n \in (\bar{F}_\epsilon)^*\}$, $m \ge 1$. From Lemma 4.6 $\mathbb{Q}^{\epsilon,*}_{x^*}(\varpi^\epsilon_m < \infty) = 1$ for all $m \in \mathbb{N}_0$. Also,

$$\begin{aligned}\mathbb{Q}^{\epsilon,*}_{x^*}(\varrho^*_\epsilon > \varpi^\epsilon_m) &= \mathbb{Q}^{\epsilon,*}_{x^*}(\varrho^*_\epsilon > \varpi^\epsilon_m, \varrho^*_\epsilon > \varpi^\epsilon_{m-1}) \\ &= \mathbb{E}^*_{x^*}(\mathbb{Q}^{\epsilon,*}_{x^*}(\varrho^*_\epsilon > \varpi^\epsilon_m \mid \mathcal{G}_{\varpi^\epsilon_{m-1}})\mathbb{I}_{\varrho^*_\epsilon > \varpi^\epsilon_{m-1}}) \\ &\le \left(\sup_{z^* \in (\bar{F}_\epsilon)^*}\mathbb{Q}^{\epsilon,*}_{z^*}(\varrho^*_\epsilon > 1)\right)\mathbb{Q}^{\epsilon,*}_{x^*}(\varrho^*_\epsilon > \varpi^\epsilon_{m-1}).\end{aligned} \tag{4.10}$$

From (4.5), (4.6) and (4.7) we get that for $\bar{x} \in \bar{F}_\epsilon$ and $x^* = (\bar{x}, 0)$

$$\mathbb{Q}^{\epsilon,*}_{x^*}(\varrho^*_\epsilon = 1) = \frac{1}{(1-\rho)}(\rho\bar{p}_\epsilon(\bar{x}, \bar{F}_\epsilon) - \rho^2\Psi_\epsilon(\bar{F}_\epsilon)) \ge \frac{\rho^2}{(1-\rho)} \equiv \lambda_0.$$

Iterating the inequalities in (4.10) we now have

$$\mathbb{Q}^{\epsilon,*}_{x^*}(\varrho^*_\epsilon > \varpi^\epsilon_m) < (1-\lambda_0)^m. \tag{4.11}$$

This shows in particular that $\mathbb{Q}^{\epsilon,*}_{x^*}(\varrho^*_\epsilon = \infty) = 0$. Furthermore, for $x^* \in (\bar{F}_\epsilon)(0)$

$$\mathbb{E}^*_{x^*}(\mathrm{e}^{\delta_5 \varrho^*_\epsilon}) = \mathbb{E}^*_{x^*}\left(\sum_{j=1}^{\infty} \mathbb{I}_{(\bar{F}_\epsilon)(0)}(\mathcal{X}_{\varpi^\epsilon_1}) \cdots \mathbb{I}_{(\bar{F}_\epsilon)(0)}(\mathcal{X}_{\varpi^\epsilon_{j-1}}) \mathbb{I}_{(\bar{F}_\epsilon)(1)}(\mathcal{X}_{\varpi^\epsilon_j}) \mathrm{e}^{\delta_5 \varpi^\epsilon_j}\right). \tag{4.12}$$

Using (4.7) it is easy to check (cf. Lemma 1 and Proposition 4 of [15])

$$E^*_{x^*}(\mathbb{I}_{(\bar{F}_\epsilon)(0)}(\mathcal{X}_{\varpi^\epsilon_k}) \mathrm{e}^{\delta_5(\varpi^\epsilon_k - \varpi^\epsilon_{k-1})} \mid \mathcal{G}_{\varpi^\epsilon_{k-1}}) \le (1-\rho)\bar{L}$$

and

$$E^*_{x^*}(\mathbb{I}_{(\bar{F}_\epsilon)(1)}(\mathcal{X}_{\varpi^\epsilon_k}) \mathrm{e}^{\delta_5(\varpi^\epsilon_k - \varpi^\epsilon_{k-1})} \mid \mathcal{G}_{\varpi^\epsilon_{k-1}}) \le \rho\bar{L}.$$

Successive conditioning in (4.12) now yields

$$\mathbb{E}^*_{x^*}(\mathrm{e}^{\delta_5 \varrho^*_\epsilon}) \le \bar{L} \sum_{j=1}^{\infty} (1-\rho)^{j-1} \rho \bar{L}^{j-1} = \frac{\bar{L}\rho}{1-(1-\rho)\bar{L}},$$

where the last equality is a consequence of (4.9). The result follows. □

It can be easily checked that for $x^*, y^* \in \bar{F}^\epsilon(1)$, $\mathbb{Q}^{\epsilon,*}_{x^*} \circ (z^1_1)^{-1} = \mathbb{Q}^{\epsilon,*}_{y^*} \circ (z^2_1)^{-1}$. From this it follows that for all $\bar{x} \in \bar{F}^\epsilon$ and $A \in \mathcal{B}(G^{\otimes\infty})$, $\mathbb{Q}^{\epsilon,*}_{\mu_{\bar{x}}}$ a.e.

$$\mathbb{Q}^{\epsilon,*}_{\mu_{\bar{x}}}(z^1_{\varrho^*_\epsilon + \cdot} \in A \mid \mathcal{G}_{\varrho^*_\epsilon}) = \mathbb{Q}^{\epsilon,*}_{\mu_{\bar{x}}}(z^2_{\varrho^*_\epsilon + \cdot} \in A \mid \mathcal{G}_{\varrho^*_\epsilon}). \tag{4.13}$$

This property is the reason for referring to the set $\bar{F}^\epsilon(1)$ as a pseudo-atom.

Coupling of continuous time processes. By our choice of r_0, $L_0 \le \gamma_2 \gamma_1^{-1} M_1$. Thus from (4.4) we can choose $M \ge L_0$ such that $\mathbb{B}_{M\epsilon^2} \subset F^1_\epsilon$. We will now prove Theorem 3.3 with this choice of M and $\varepsilon_2 = \theta_0$, $\delta_2 = \frac{\delta_5}{L}$. Fix $\epsilon \in (0, \varepsilon_2)$ and $x_1, x_2 \in \mathbb{B}_{M\epsilon^2}$. The main step in the proof is the construction, for each $\epsilon \in (0, \varepsilon_2)$, of a filtered probability space $\Sigma^\epsilon = (\Omega^\epsilon, \mathcal{G}^\epsilon, (\mathcal{G}^\epsilon_t)_{t\ge 0}, \hat{\mathbb{P}}^\epsilon)$ on which are given continuous adapted G valued processes $\hat{Z}^1, \hat{Z}^2$ and sequence of H^* valued random variables $\hat{\mathcal{X}} \equiv (\hat{\mathcal{X}}_n)_{n\in\mathbb{N}_0} \equiv (\hat{z}^1_n, \hat{z}^2_n, \hat{i}_n)_{n\in\mathbb{N}_0}$ such that with $\bar{x} = (x_1, x_2)$:

(a) $\hat{\mathbb{P}}^\epsilon \circ (\hat{\mathcal{X}})^{-1} = \mathbb{Q}^{\epsilon,*}_{\mu_{\bar{x}}}$.

(b) $\hat{\mathbb{P}}^\epsilon \circ (\hat{Z}^1, \hat{Z}^2)^{-1} = \tilde{\mathbb{P}}^\epsilon_{x_1,x_2} \circ (Z^1, Z^2)^{-1}$.

(c) $\hat{\mathbb{P}}^\epsilon(\hat{Z}^i(n\Delta_\epsilon) = \hat{z}^i_n) = 1$, $i = 1, 2$, $n \in \mathbb{N}_0$.

Once such a construction is completed, Theorem 3.3 will follow on setting $\varrho_\epsilon = (\hat{\varrho}^*_\epsilon + 1)\Delta_\epsilon$, where $\hat{\varrho}^*_\epsilon = \inf\{n \in \mathbb{N}_0 : \hat{\mathcal{X}}_n \in \bar{F}_\epsilon(1)\}$.

Construction of Σ^ϵ. Define $\Omega^\epsilon = (H^*)^{\otimes\infty} \times \mathbb{C} \times \mathbb{C}$, where $\mathbb{C} = C([0,\infty) : G)$. To define $\hat{\mathbb{P}}^\epsilon$ we proceed as follows.

• *Law of the Split Chain:* Let $\hat{\mu}^\epsilon = \mathbb{Q}^{\epsilon,*}_{\mu_{\bar{x}}}$. Note that $\hat{\mu}^\epsilon \in \mathcal{P}((H^*)^{\otimes\infty})$.

• *Disintegrating* $\hat{\mu}^\epsilon$: Write $(H^*)^{\otimes\infty} = H^{\otimes\infty} \times \{0,1\}^{\otimes\infty}$. Disintegrate $\hat{\mu}^\epsilon$ as

$$\hat{\mu}^\epsilon(d\beta, d\alpha) = \hat{\mu}^\epsilon_2(\beta, d\alpha)\hat{\mu}^\epsilon_1(d\beta), \quad \beta \in H^{\otimes\infty}, \ \alpha \in \{0,1\}^{\otimes\infty}.$$

From Lemma 4.4, $\hat{\mu}^\epsilon_1 = \mathbb{Q}^\epsilon_{\bar{x}}$.

• *Describing the marginal distribution on* $H^{\otimes\infty}$: Write $H^{\otimes\infty} = H \times H \cdots$ and recalling the definition of $\bar{p}_\epsilon$ introduced above Lemma 4.3 we can disintegrate $\hat{\mu}^\epsilon_1$ as

$$\hat{\mu}^\epsilon_1(d\beta) = \delta_{\bar{x}}(dr_1)\bar{p}_\epsilon(r_1, dr_2)\bar{p}_\epsilon(r_2, dr_3)\cdots, \quad \beta = (r_1, r_2, \ldots).$$

• *Law of a Δ_ϵ segment of the continuous process:* Let $\mathbb{C}^\epsilon = C([0, \Delta_\epsilon] : H)$. Recall $\bar{Z}$ introduced above Lemma 4.2. Denote the process $\{\bar{Z}(t)\}_{0 \le t \le \Delta_\epsilon}$ by $\bar{Z}_{[0,\Delta_\epsilon]}$. Note that $\bar{Z}_{[0,\Delta_\epsilon]}$ is a $\mathbb{C}^\epsilon$ valued random variable. For $\bar{y} = (y_1, y_2) \in H$, let $\hat{\mathbb{Q}}^\epsilon_{\bar{y}} = \tilde{\mathbb{P}}^\epsilon_{y_1,y_2} \circ (\bar{Z}_{[0,\Delta_\epsilon]})^{-1}$.

• Let $\mathcal{U}^\epsilon = \mathbb{C}^\epsilon \times H$ and define $\hat{\psi}^\epsilon : \mathbb{C}^\epsilon \to \mathcal{U}^\epsilon$ as $\hat{\psi}^\epsilon(\omega) = (\omega, \omega(\Delta_\epsilon))$. Define $\hat{\mathcal{R}}^\epsilon_{\bar{y}} = \hat{\mathbb{Q}}^\epsilon_{\bar{y}} \circ (\hat{\psi}^\epsilon)^{-1}$. Then, $\hat{\mathcal{R}}^\epsilon_{\bar{y}} \in \mathcal{P}(\mathcal{U}^\epsilon)$. Disintegrate $\hat{\mathcal{R}}^\epsilon_{\bar{y}}$ as

$$\hat{\mathcal{R}}^\epsilon_{\bar{y}}(d\theta, d\bar{z}) = \hat{\mathcal{T}}^\epsilon_{\bar{y},\bar{z}}(d\theta)\hat{\nu}^\epsilon_{\bar{y}}(d\bar{z}), \quad \theta \in \mathbb{C}^\epsilon, \ \bar{z} \in H.$$

• *Describing the law of the infinite sequence of Δ_ϵ segments:* Define for $n \ge 1$, $\hat{\eta}^\epsilon_n \in \mathcal{P}(H \times (\mathcal{U}^\epsilon)^{\otimes n})$ as

$$\begin{aligned}\hat{\eta}^\epsilon_n(df_1, dr_1, \ldots, df_n, dr_n, dr_{n+1}) = {}& \delta_{\bar{x}}(dr_1)\bar{p}_\epsilon(r_1, dr_2)\hat{\mathcal{T}}^\epsilon_{r_1,r_2}(df_1)\bar{p}_\epsilon(r_2, dr_3)\hat{\mathcal{T}}^\epsilon_{r_2,r_3}(df_2) \\ & \cdots \bar{p}_\epsilon(r_n, dr_{n+1})\hat{\mathcal{T}}^\epsilon_{r_n,r_{n+1}}(df_n), \ r_1, \ldots, r_{n+1} \in H, \\ & f_1, \ldots, f_n \in \mathbb{C}^\epsilon.\end{aligned}$$

By Kolmogorov's consistency theorem the sequence $\{\hat{\eta}^\epsilon_n\}_{n\in\mathbb{N}}$ determines uniquely a $\hat{\eta}^\epsilon \in \mathcal{P}((\mathcal{U}^\epsilon)^{\otimes\infty})$ consistent with the sequence.

• *Pasting the segments together:* Let $\mathbb{U}^\epsilon \subset (\mathcal{U}^\epsilon)^{\otimes\infty}$ be the collection of all $(f_1, r_1, f_2, r_2, \ldots) \in (\mathcal{U}^\epsilon)^{\otimes\infty}$ with the property $f_l(\Delta_\epsilon) = f_{l+1}(0) = r_{l+1}$ for all $l \in \mathbb{N}$. By construction, $\hat{\eta}^\epsilon(\mathbb{U}^\epsilon) = 1$. Define $\hat{\phi}^\epsilon : \mathbb{U}^\epsilon \to H^{\otimes\infty} \times \mathbb{C} \times \mathbb{C}$ as

$$\hat{\phi}^\epsilon((f_1, r_1), \ldots) = ((r_1, r_2, \ldots), h_1, h_2)$$

with $(h_1(t), h_2(t)) = f_l(t)$, if $t \in [(l-1)\Delta_\epsilon, l\Delta_\epsilon)$. Denote the measure $\hat{\eta}^\epsilon \circ (\hat{\phi}^\epsilon)^{-1}$ on $H^{\otimes\infty} \times \mathbb{C} \times \mathbb{C}$ by $\hat{\pi}^\epsilon$.

• Disintegrate $\hat{\pi}^\epsilon$ as

$$\hat{\pi}^\epsilon(da, db, dc) = \hat{\pi}^\epsilon_2(a, db, dc)\hat{\pi}^\epsilon_1(da), \quad a \in H^{\otimes\infty}, \ b, c \in \mathbb{C}.$$

By construction $\hat{\pi}^\epsilon_1 = \hat{\mu}^\epsilon_1 = \mathbb{Q}^\epsilon_{\bar{x}}$.

• *Defining the continuous time analog of split chain:* Finally define $\hat{\mathbb{P}}^\epsilon$ on $(H^*)^{\otimes\infty} \times \mathbb{C} \times \mathbb{C} = \Omega^\epsilon$ as

$$\begin{aligned}&\hat{\mathbb{P}}^\epsilon(da, db, df_1, df_2) = \hat{\mu}^\epsilon_2(a, db)\hat{\pi}^\epsilon_2(a, df_1, df_2)\hat{\mu}^\epsilon_1(da), \\ &\quad a \in H^\infty, \ b \in \{0,1\}^\infty, \ f_1, f_2 \in \mathbb{C}.\end{aligned}$$

Denote by $\hat{Z} = (\hat{Z}^1, \hat{Z}^2)$ and $\hat{\chi} \equiv (\hat{\chi}_n)_{n\in\mathbb{N}_0} \equiv (\hat{z}^1_n, \hat{z}^2_n, \hat{i}_n)_{n\in\mathbb{N}_0}$ the canonical processes and sequences on $(H^*)^{\otimes\infty} \times \mathbb{C} \times \mathbb{C}$:

$$\hat{Z}(t)(\omega) = (f_1(t), f_2(t)), \hat{\chi}_n(\omega) = (\hat{z}^1_n, \hat{z}^2_n, \hat{i}_n) = (a^1_n, a^2_n, b_n), \quad n \in \mathbb{N}_0, t \ge 0,$$

where $\omega = (a, b, f_1, f_2)$, $a = (a^1, a^2)$, $a^i = (a^i_n)_{n\in\mathbb{N}_0} \in G^{\otimes\infty}$, $f_i \in C([0,\infty) : G)$, $i = 1, 2$, $b = (b_n)_{n\in\mathbb{N}_0} \in [0,1]^{\otimes\infty}$. Also, let $\mathcal{G}^\epsilon_t = \sigma\{\hat{Z}(s), \hat{i}_{\lfloor \frac{s}{\Delta_\epsilon} \rfloor \Delta_\epsilon} : s \le t\}$.

By construction the desired properties (a)–(c) are satisfied. Moreover, the construction and (4.13) ensure that with $\varrho_\epsilon = (\hat{\varrho}^*_\epsilon + 1)\Delta_\epsilon$, for all $A \in \mathcal{B}(\mathbb{C})$,

$$\hat{\mathbb{P}}^\epsilon(\hat{Z}^1(\varrho_\epsilon + \cdot) \in A \mid \mathcal{G}^\epsilon_{\varrho_\epsilon}) = \hat{\mathbb{P}}^\epsilon(\hat{Z}^2(\varrho_\epsilon + \cdot) \in A \mid \mathcal{G}^\epsilon_{\varrho_\epsilon}).$$

An application of Lemma 4.7 now completes the proof of Theorem 3.3.

5. Invariant measure asymptotics

From Theorem 2.2 of [2] it follows that under the standing assumptions of this paper (i.e. Conditions 2.1–2.4), for each $\epsilon > 0$, the Markov process $(Z, \{\mathbb{P}_x^\epsilon\}_{x\in G})$ has a unique invariant measure denoted as μ^ϵ. The measure μ^ϵ has a nonvanishing density on G (cf. Lemma 5.7 of [12]). In this section we study the asymptotic properties of μ^ϵ as $\epsilon \to 0$. We begin with the following basic convergence result. Let δ_0 denote the Dirac probability measure concentrated at $0 \in \mathbb{R}^k$.

Theorem 5.1. *The collection* $\{\mu^\epsilon\}_{\epsilon\in[0,1]}$ *is tight and* μ^ϵ *converges weakly to* δ_0 *as* $\epsilon \to 0$.

Proof. From Lemma 4.1 it follows that $\int_G \exp\{\frac{\alpha T(x)}{\Delta_\epsilon}\}\mu^\epsilon(\mathrm{d}x) \le m_1/\beta$. The tightness of $\{\mu^\epsilon\}_{\epsilon\in[0,1]}$ is immediate from this estimate. Let μ_0 be a weak limit point along a subsequence $\epsilon_n \to 0$. Fix $\upsilon_0 \in (0, 1)$ and choose a compact set $K_{\upsilon_0} \subset G$ such that $\sup_{\epsilon\in[0,1]} \mu^\epsilon(K^c_{\upsilon_0}) \le \upsilon_0$. Then

$$\mu_0(K^c_{\upsilon_0}) \le \upsilon_0. \tag{5.1}$$

Let $s_0 = \sup_{x\in K_{\upsilon_0}} T(x)$. Note that for $t \ge s_0$, $z(t) = 0$ for all $z \in \mathcal{A}(x)$ and all $x \in K_{\upsilon_0}$. Denote the measure induced by $\{Z(t), t \in [0, s_0]\}$, under $\mathbb{P}^\epsilon_{\mu^\epsilon} = \int_G \mathbb{P}^\epsilon_x \mu^\epsilon(\mathrm{d}x)$, on $C([0, s_0] : G)$ by $\mathbb{Q}^\epsilon$. It is easily checked that $\{\mathbb{Q}^\epsilon, \epsilon \in (0, 1]\}$ is tight and so, along some subsequence, $\mathbb{Q}^{\epsilon_n}$ converges weakly to some $\mathbb{Q}_0 \in \mathcal{P}(C([0, s_0] : G))$. Furthermore, $\mathbb{Q}_0 \circ (\tilde{Z}(s_0))^{-1} = \mathbb{Q}_0 \circ (\tilde{Z}(0))^{-1}$ and

$$\tilde{Z}(t) = \Gamma\left(\tilde{Z}(0) + \int_0^\cdot b(\tilde{Z}(s))\mathrm{d}s\right)(t), \quad t \in [0, s_0],\ \mathbb{Q}_0 \text{ a.e.,}$$

where $\tilde{Z}$ is the canonical process on $C([0, s_0] : G)$. Next note that

$$\begin{aligned}\mu_0(K_{\upsilon_0} \setminus \{0\}) = \mathbb{Q}_0(\tilde{Z}(s_0) \in K_{\upsilon_0} \setminus \{0\}) &= \mathbb{Q}_0(\tilde{Z}(s_0) \in K_{\upsilon_0} \setminus \{0\} \mid \tilde{Z}(0) \in K_{\upsilon_0})\mu_0(K_{\upsilon_0}) \\ &\quad + \mathbb{Q}_0(\tilde{Z}(s_0) \in K_{\upsilon_0} \setminus \{0\} \mid \tilde{Z}(0) \in K^c_{\upsilon_0})\mu_0(K^c_{\upsilon_0}).\end{aligned}$$

The first term on the right side above is zero while the second is bounded by υ_0. Combining this observation with (5.1) we have $\mu_0(G \setminus \{0\}) \le 2\upsilon_0$. The result follows on sending $\upsilon_0 \to 0$. $\square$

The following is the analog of Theorem 4.3 of [20]. Since the assumptions of the cited theorem (see Assumption A on page 128 of [20]) are somewhat different from those made in this work, we sketch a proof in Section 6. Let B be a bounded open set in G such that $\partial B = \partial \bar{B}$.

Theorem 5.2. $\epsilon^2 \log \mu^\epsilon(B)$ *converges to* $-\inf_{x\in B} V(x)$*, as* $\epsilon \to 0$.

6. Some proofs

Sketch of Proof of Theorem 2.1. We will apply Theorem 5 of [10] (see also [8]). For $x \in G$ and $u \in \mathrm{AC}([0, T] : \mathbb{R}^k)$ such that $\int_0^T |\dot{u}(s)|^2\mathrm{d}s < \infty$, let $\mathcal{G}^0(x, u) = \phi$, where ϕ is the unique solution of

$$\phi(t) = \Gamma\left(x + \int_0^\cdot b(\phi(s))\mathrm{d}s + \int_0^\cdot \sigma(\phi(s))\dot{u}(s)\mathrm{d}s\right), \quad t \ge 0.$$

Let $x^\epsilon \in G$ be such that $x^\epsilon \to x$ as $\epsilon \to 0$. Also, let $\{h^\epsilon\}_{\epsilon>0}$ be a collection of $\mathbb{R}^k$ valued $\{\mathcal{F}_t\}$ predictable processes such that, for some $M \in (0, \infty)$, $\int_0^T |h^\epsilon(s, \omega)|^2\mathrm{d}s \le M$, for all $\epsilon > 0$ and $\omega \in \mathcal{F}$. For each $\epsilon > 0$, h^ϵ can be regarded as an $\mathbb{H}_M$ valued random variable where $\mathbb{H}_M = \{e \in L^2([0, T] : \mathbb{R}^k) : \int_0^T |e(s)|^2\mathrm{d}s \le M\}$ endowed with the topology inherited from the weak topology on $L^2([0, T] : \mathbb{R}^k)$. Suppose h^ϵ converges in distribution (under $\mathbb{P}^\epsilon_{x^\epsilon}$) to some

$\mathbb{H}_M$ valued random variable h. Let $\tilde{X}^\epsilon$ solve, $\mathbb{P}^\epsilon_{x^\epsilon}$ a.e.,

$$\tilde{X}^\epsilon(t) = \Gamma\left(x^\epsilon + \int_0^\cdot b(\tilde{X}^\epsilon(s))\mathrm{d}s + \int_0^\cdot \sigma(\tilde{X}^\epsilon(s))h^\epsilon(s)\mathrm{d}s + \epsilon\int_0^\cdot \sigma(\tilde{X}^\epsilon(s))dB(s)\right),\quad t \geq 0.$$

Using the Lipschitz property of the Skorokhod map and of the coefficients b, σ, it is easy to verify that $\tilde{X}^\epsilon$ (under $\mathbb{P}^\epsilon_{x^\epsilon}$) converges weakly to $\mathcal{G}^0(x, u)$, where $u(t) = \int_0^t h(s)\mathrm{d}s$, $t \in [0, T]$. Due to unique pathwise solvability of (1.4), one can write $\tilde{X}^\epsilon = \mathcal{G}^\epsilon(x^\epsilon, \epsilon B + \int_0^\cdot h^\epsilon(s)\mathrm{d}s)$, where $\mathcal{G}^\epsilon$ is a measurable map from $G \times C([0, T] : \mathbb{R}^k)$ to $C([0, T] : G)$, such that $\mathcal{G}^\epsilon(x^\epsilon, \epsilon B) \equiv X^\epsilon$ is the unique solution of (2.4) when $x = x^\epsilon$. The result is now an immediate consequence of Theorem 5 of [10]. $\square$

Sketch of Theorem 3.1. We first consider (3.2). Let $x_0 \in \arg\min\{V(x) : x \in \partial B\}$. For $h > 0$, let $F_h = \{x \in B^c : \mathrm{dist}(x, B) \geq h\}$. Using the fact that $\partial B = \partial \bar{B}$ it follows that for every $\theta > 0$, there exist $x_\theta \in B^c$ and $h > 0$ such that $|x_\theta - x_0| < \theta$ and $x_\theta \in F_h$. Thus, for every $\gamma > 0$, by a straightforward pasting argument, one can find $h, \mu, T_2 > 0$, and, for every $x \in \mathbb{B}_\mu$, a pair $(\phi^x, \psi^x) \in \mathrm{AC}([0,\infty) : G) \times \mathrm{AC}([0,\infty) : \mathbb{R}^k)$ such that the following hold: (i) $\mathbb{B}_\mu \subset B$, (ii) for every $x \in \mathbb{B}_\mu$, $\phi^x(0) = x$ and $s(x) = \inf\{t > 0 : \phi^x(t) \in F_h\} \leq T_2$, (iii) for every $x \in \mathbb{B}_\mu$, $\phi^x = \Gamma(\psi^x)$ and $j_{s(x)}(\phi^x, \psi^x) < V_0 + \gamma/4$.

From the uniform LDP in Theorem 2.1, we can find $\epsilon_1 > 0$ such that for all $\epsilon \in (0, \epsilon_1)$,

$$\inf_{y\in\mathbb{B}_\mu} \mathbb{P}^\epsilon_y(\tau < T_2) \geq \inf_{y\in\mathbb{B}_\mu} \mathbb{P}^\epsilon_y\left\{\sup_{0\leq t\leq s(y)} |Z(t) - \phi^y(t)| < h\right\} \geq \exp(-\epsilon^{-2}(V_0 + \gamma/2)).$$

Recall the function ξ_x introduced in (3.1). Since B is bounded, using Lemma 2.1 we can find a $T_1 \in (0, \infty)$ such that for all $x \in B \bigcup \partial B$

$$\xi_x(t) \in \mathbb{B}_{\mu/2}, \quad \text{for all } t \geq T_1. \tag{6.1}$$

Let $\sigma = \min\{t : Z(t) \in \mathbb{B}_\mu\}$. Then from (6.1), the Lipschitz property of the SM and boundedness of the diffusion coefficient, it follows that $\lim_{\epsilon\to 0}\sup_{x\in\bar{B}} \mathbb{P}^\epsilon_x(\sigma \geq T_1) = 0$. Using the strong Markov property of $(Z, \{\mathbb{P}^\epsilon_x\})$, it now follows that, for some $\epsilon_2 \leq \epsilon_1$ and for all $\epsilon < \epsilon_2$,

$$\mathbb{P}^\epsilon_x(\tau < T_1 + T_2) \geq \mathbb{P}^\epsilon_x(\sigma < T_1)\exp(-\epsilon^{-2}(V_0 + \gamma/2)) \geq \frac{1}{2}\exp(-\epsilon^{-2}(V_0 + \gamma/2)).$$

Following [20] (argument on page 125) we now have that

$$\mathbb{E}^\epsilon_x(\tau) \leq 2(T_1 + T_2)\exp\{\epsilon^{-2}(V_0 + \gamma)\}.$$

Sending $\epsilon \to 0$, (3.2) follows. Proof of (3.3) is immediate from (3.2) by an application of Markov's inequality (cf. Theorem 4.4.2 of [20]).

Now consider (3.4). Following [20] define $\varpi = \partial\mathbb{B}_{\mu/2}$, $\varpi_1 = \partial\mathbb{B}_\mu$ and define stopping times θ_1, β_1, as $\beta_1 = \inf\{t \geq 0 : Z(t) \in \varpi_1\}$ and $\theta_1 = \inf\{t > \beta_1 : Z(t) \in \varpi \text{ or } Z(t) \in B^c\}$. Then for $x \in \varpi$ and $T > 0$,

$$\mathbb{P}^\epsilon_x(Z(\theta_1) \in B^c) \leq \max_{y\in\varpi_1}\{\mathbb{P}^\epsilon_y(\theta_1 = \tau < T) + \mathbb{P}^\epsilon_y(\theta_1 = \tau \geq T)\}.$$

Using Lemma 3.3, for some $T > 0$ and $\epsilon_1 > 0$,

$$\mathbb{P}^\epsilon_y(\theta_1 \geq T) \leq \exp(-\epsilon^{-2}V_0), \quad \text{for all } \epsilon < \epsilon_1 \text{ and } y \in \varpi_1. \tag{6.2}$$

Fix $\gamma > 0$. Recalling the definition of V_0, there is a $0 < \epsilon_2 \leq \epsilon_1$ (see p. 126 [20]) such that for all $y \in \varpi_1$ (with μ sufficiently small),

$$\mathbb{P}^\epsilon_y(\tau = \theta_1 < T) \leq \exp(-\epsilon^{-2}(V_0 - \gamma)).$$

Following the argument on page 126 of [20] one then has that there is a $\kappa > 0$ such that for all $x \in \varpi$ and $\epsilon < \epsilon_2$,

$$\mathbb{E}^\epsilon_x(\tau) \geq \kappa \exp(\epsilon^{-2}(V_0 - \gamma)).$$

Sending $\epsilon \to 0$, (3.4) follows for $x \in \mathbb{B}_{\mu/2}$. For a general $x \in B_0$, using the strong Markov property,

$$\mathbb{E}^\epsilon_x(\tau) \geq \kappa \exp(-\epsilon^{-2}(V_0 - \gamma))\mathbb{P}^\epsilon_x(\tau > \tilde{\tau}),$$

where $\tilde{\tau} = \inf\{t : Z(t) \in \varpi\}$. Finally using the definition of B_0, we have that

$$\lim_{\epsilon \to 0} \mathbb{P}^\epsilon_x(\tau > \tilde{\tau}) = 1, \quad \text{for all } x \in B_0. \tag{6.3}$$

Using this property in the above equation and sending $\epsilon \to 0$, we have (3.4) for any $x \in B_0$. The proof of (3.5) follows exactly as the proof of Theorem 4.4.2 of [20] upon making use of (6.3) once again. $\square$

Sketch of Proof of Theorem 5.2. Suppose first that $V_1 > 0$. Then $\text{dist}(0, B) = \kappa_1 \in (0, \infty)$. Let $\kappa_2 \in (0, \infty)$ be such that $\text{dist}(\mathbb{B}_{\kappa_2}, B) > 0$. Fix $0 < \theta_1 < \theta_2 < \kappa_2$. We will choose θ_1, θ_2 suitably small later in the proof. For $\theta \in (0, \infty)$, let $\varpi(\theta) = \partial\mathbb{B}_\theta$. Define stopping times σ_0, τ_1 as $\sigma_0 = \inf\{t : Z(t) \in \varpi(\theta_2)\}$ and $\tau_1 = \inf\{t \geq \sigma_0 : Z(t) \in \varpi(\theta_1)\}$. Then for a suitable $\ell^\epsilon \in \mathcal{P}(\varpi(\theta_1))$,

$$\mu^\epsilon(B) = c_\epsilon \int_{\varpi(\theta_1)} \mathbb{E}^\epsilon_x \left(\int_0^{\tau_1} \mathbb{I}_B(Z(s))\mathrm{d}s \right) \ell^\epsilon(\mathrm{d}x),$$

where $c_\epsilon^{-1} = \int_{\varpi(\theta_1)} \mathbb{E}^\epsilon_x(\tau_1)\ell^\epsilon(\mathrm{d}x)$. Let $\tau_B = \inf\{t : Z(t) \in B\}$. Then

$$\mu^\epsilon(B) \leq c_\epsilon \max_{x \in \varpi(\theta_1)} \mathbb{P}^\epsilon_x(\tau_B < \tau_1) \max_{y \in \partial B} \mathbb{E}^\epsilon_y(\tau_1).$$

From Lemma 3.3, we can find $s_1 > 0$, $\kappa_3 \in (0, \kappa_2)$ and $\varepsilon_1 \in (0, 1)$ such that whenever θ_1, $\theta_2 \in (0, \kappa_3)$ and $\epsilon \in (0, \varepsilon_1)$

$$\sup_{y \in \varpi(\theta_2)} \mathbb{P}^\epsilon_y(\tau_1 > s_1) \leq \exp(-\epsilon^{-2}V_1).$$

Fix $\gamma \in (0, 1)$. Then, one can find $\kappa_5 \in (0, \kappa_4)$ and $\varepsilon_2 \in (0, \varepsilon_1)$ such that for all $\epsilon \in (0, \varepsilon_2)$ and $\theta \in (0, \kappa_5)$

$$\sup_{y \in \varpi(\theta)} \mathbb{P}^\epsilon_y(\tau_B \leq s_1) \leq \exp(-\epsilon^{-2}(V_1 - \gamma)).$$

Thus for $0 < \theta_1 < \theta_2 \leq \kappa_5$,

$$\begin{aligned}\sup_{x \in \varpi(\theta_1)} \mathbb{P}^\epsilon_x(\tau_B < \tau_1) &\leq \sup_{y \in \varpi(\theta_2)} \mathbb{P}^\epsilon_y(\tau_B \leq s_1) + \sup_{y \in \varpi(\theta_2)} \mathbb{P}^\epsilon_y(\tau_1 > s_1) \\ &\leq 2\exp(-\epsilon^{-2}(V_1 - \gamma)).\end{aligned}$$

Fix a choice of θ_1, θ_2 such that any path ξ_x starting from $\varpi(\theta_1)$ keeps a positive distance from $\varpi(\theta_2)$. Using Theorem 3.1(ii) (also Lemma 3.2), we can find $\varepsilon_3 \in (0, \varepsilon_2)$ and $\gamma_0 > 0$ such that for all $\epsilon \in (0, \varepsilon_3)$

$$\inf_{x \in \varpi(\theta_1)} \epsilon^2 \log \mathbb{E}_x^\epsilon(\tau_1) \geq \inf_{x \in \varpi(\theta_1)} \epsilon^2 \log \mathbb{E}_x^\epsilon(\sigma_0) \geq \epsilon^2 \log(\gamma_0).$$

Using Lemma 3.3 we can find $\varepsilon_4 \in (0, \varepsilon_3)$ and $a_1 \in (0, \infty)$ such that for all $\epsilon \in (0, \varepsilon_3)$, $\sup_{x \in \partial B} \mathbb{E}_x^\epsilon(\tau_1) \leq a_1$. Combining these estimates we have $\epsilon^2 \log \mu^\epsilon(B) \leq (\log(2a_1) - \log \gamma_0) \epsilon^2 - V_1 + \gamma$. Since $\gamma > 0$ is arbitrary, we obtain $\limsup_{\epsilon \to 0} \epsilon^2 \log \mu^\epsilon(B) \leq -V_1$.

Now we prove the reverse inequality. Fix $\gamma \in (0, 1)$. Choose $\beta > 0$ such that $B_\beta = \{x \in B | \mathrm{dist}(x, \partial B) > \beta\}$ satisfies $\inf_{x \in B_\beta} V(x) < V_1 + \gamma/2$. Then for some $a_2 \in (0, \infty)$

$$\inf_{x \in B_\beta} \mathbb{E}_x^\epsilon \int_0^{\tau_1} \mathbb{I}_B(Z(s)) \mathrm{d}s \geq a_2.$$

Let $\sigma_\beta = \inf\{t : Z(t) \in B_\beta\}$. Choose $0 < \theta_1 < \theta_2 < \kappa_2$, $\varepsilon > 0$, $a_3 \in (0, \infty)$ such that for all $\epsilon \in (0, \varepsilon)$

$$\inf_{x \in \varpi(\theta_1)} \mathbb{P}_x^\epsilon(\sigma_\beta < \tau_1) \geq \exp\{-\epsilon^{-2}(V_1 + \gamma)\},$$

and

$$\sup_{x \in \varpi(\theta_1)} \epsilon^2 \log \mathbb{E}_x^\epsilon(\sigma_0) \leq \gamma, \qquad \sup_{x \in \varpi(\theta_2)} \log \mathbb{E}_x^\epsilon(\tau_1) \leq a_3.$$

Combining the above estimates we have for all $\epsilon \in (0, \varepsilon)$,

$$\epsilon^2 \log \mu^\epsilon(B) \geq -2\gamma - ((2 \log 2) - \log a_2 + a_3)\epsilon^2 - V_1.$$

Since $\gamma \in (0, 1)$ is arbitrary, we have, $\liminf_{\epsilon \to 0} \epsilon^2 \log \mu^\epsilon(B) \geq -V_1$. This proves the result when $V_1 > 0$.

Consider now the case $V_1 = 0$. In this case $0 \in \bar{B}$. If $0 \in B$, the result follows from Theorem 5.1. If $0 \in \partial B$, from continuity of V we can find, for each $h \in (0, 1)$, an open set $B_1 \subset B$ such that $0 < \inf_{x \in B_1} V(x) \leq h$. From the first part of the proof

$$\liminf_{\epsilon \to 0} \epsilon^2 \log \mu^\epsilon(B) \geq \liminf_{\epsilon \to 0} \epsilon^2 \log \mu^\epsilon(B_1) = -\inf_{x \in B_1} V(x) \geq -h.$$

Since $h \in (0, 1)$ is arbitrary, the result follows on sending $h \to 0$. □

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